Unit 9 · Inverse Circular Functions

HARDER PRACTICE QUESTIONS

Full marks 29

Write your answers on separate paper. Show your working. Angles are in radians unless degrees are given; use $\pi \approx 3.1416$ where you need a number.

1 Back into the branch [5]

a. Find $\cos^{-1}(\cos 10)$, $\tan^{-1}(\tan 10)$ and $\sin^{-1}(\sin 5)$, each in terms of π.

b. Sketch $y = \cos^{-1}(\cos x)$ for $0 \le x \le 4\pi$, and use it to explain your first answer.

2 The other side of an identity [4]

a. Show that for $x > 1$, $\sin^{-1}\dfrac{2x}{1+x^2} = \pi - 2\tan^{-1} x$.

b. Check it at $x = \sqrt{3}$.

c. What is $\sin^{-1}\dfrac{2x}{1+x^2}$ for $x < -1$? Give a reason.

3 A root too many [6]

a. Solve $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\dfrac{8}{31}$.

b. For which numbers c has $\tan^{-1}(x+1) + \tan^{-1}(x-1) = c$ a solution, and how many? Use the graph of $\tan^{-1}$ to say, and explain from it, before solving anything, why a. could have only one root.

4 Three inverse cosines [5]

a. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$. Check it with $x = y = z = \dfrac{1}{2}$, and with $x = y = 0$.

b. Rojan says the converse is true too: if $x^2 + y^2 + z^2 + 2xyz = 1$, the three inverse cosines add up to π. Test him with $x = y = \dfrac{1}{2}$, and say which step of your proof can't be turned round.

5 Designing a formula for π [4]

You know that $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{4}$.

a. Find x with $\tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1}{5} + \tan^{-1} x$, and hence show that $\dfrac{\pi}{4} = \tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{8}$.

b. Ujwal says the new formula will give π faster with Madhava's series than the old one, because its fractions are smaller. Is he right? Use how fast each series' terms shrink.

6 The best place to stand [5]

A notice board 3 m tall is fixed on a wall with its bottom edge 1 m above Sushant's eye level. Standing x m from the wall, he sees the board at an angle θ (between the lines to its top and bottom edges).

a. Show that $\theta = \tan^{-1}\dfrac{4}{x} - \tan^{-1}\dfrac{1}{x} = \tan^{-1}\dfrac{3x}{x^2+4}$.

b. Can θ ever be $45°$? Use the discriminant.

c. By writing $\frac{3x}{x^2+4} = \frac{3}{x+\frac{4}{x}}$, find where he should stand to make θ as large as possible, and the angle in degrees.