Unit 9 · Inverse Circular Functions

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 29

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Back into the branch

HA · 5

ANSWER

Tests: $f^{-1}(f(\theta))$ is the angle in the branch with the same ratio; the tempting 10 (or the right size with the wrong sign).

a. $\cos^{-1}(\cos 10)$: the angle in $[0, \pi]$ with the same cosine. $3\pi \approx 9.42 < 10 < 4\pi \approx 12.57$, and $\cos 10 = \cos(4\pi - 10)$, with $4\pi - 10 \approx 2.566$ in $[0, \pi]$: $4\pi - 10$.

$\tan^{-1}(\tan 10)$: the tangent has period π, so subtract 3π: $10 - 3\pi \approx 0.575$, in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\sin^{-1}(\sin 5)$: $\sin 5 = \sin(5 - 2\pi)$, and $5 - 2\pi \approx -1.283$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$: $5 - 2\pi$. (The size 1.283 with a plus sign would have the wrong sine: $\sin 5 < 0$.)

b. A zigzag between 0 and π: it rises as $y = x$ from 0 to π, falls as $y = 2\pi - x$ to 0 at 2π, rises as $y = x - 2\pi$ to π at 3π, and falls as $y = 4\pi - x$ to 0 at 4π. At $x = 10$ (between 3π and 4π) the falling piece gives $4\pi - 10$.

MARKING SCHEME

✓ a. $4\pi - 10$.

✓ $10 - 3\pi$.

✓ $5 - 2\pi$, negative, with the reason.

✓ b. The zigzag between 0 and π, corners at multiples of π.

✓ $x = 10$ is on the piece $y = 4\pi - x$.

2 The other side of an identity

HA · 4

ANSWER

Tests: an identity holds only where its multiple angle stays in the branch; working out what replaces it outside.

a. Let $\theta = \tan^{-1} x$. For $x > 1$, $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, so $\frac{\pi}{2} < 2\theta < \pi$: outside $\sin^{-1}$'s branch. Still $\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta} = \frac{2x}{1 + x^2}$. The angle $\pi - 2\theta$ has the same sine and lies in $\left(0, \frac{\pi}{2}\right)$, inside the branch, so $\sin^{-1}\frac{2x}{1 + x^2} = \pi - 2\theta$.

b. $\frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$, so the left side is $\frac{\pi}{3}$; $\pi - 2 \cdot \frac{\pi}{3} = \frac{\pi}{3}$. ✓

c. $-\pi - 2\tan^{-1} x$. Both sides of a. are odd in x ($\sin^{-1}$ and $\tan^{-1}$ are odd): for $x < -1$, $-x > 1$, so $\sin^{-1}\frac{2x}{1 + x^2} = -\sin^{-1}\frac{-2x}{1 + x^2} = -\left(\pi - 2\tan^{-1}(-x)\right) = -\pi - 2\tan^{-1} x$.

MARKING SCHEME

✓ a. 2θ is between $\frac{\pi}{2}$ and π, outside the branch.

✓ $\sin(\pi - 2\theta) = \sin 2\theta$, with $\pi - 2\theta$ in the branch.

✓ b. Both sides $\frac{\pi}{3}$.

✓ c. $-\pi - 2\tan^{-1} x$, by oddness.

3 A root too many

ANSWER

Tests: taking tangents can add a root; the condition $xy < 1$ decides; the left side's graph shows in advance how many roots there are.

a.

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking

Tangent of both sides: $\dfrac{2x}{1-(x^2-1)} = \dfrac{2x}{2-x^2} = \dfrac{8}{31}$, so $62x = 16 - 8x^2$, $4x^2 + 31x - 8 = 0$, $(4x-1)(x+8) = 0$: $x = \dfrac{1}{4}$ or $x = -8$.

Check. $x = \dfrac{1}{4}$: the values $\dfrac{5}{4}$ and $-\dfrac{3}{4}$ have product $-\dfrac{15}{16} < 1$, so the sum is $\tan^{-1}\dfrac{8}{31}$. ✓ $x = -8$: the values -7 and -9, both negative with product $63 > 1$, so the sum is $\tan^{-1}\dfrac{8}{31} - \pi$, a negative angle. ✗

The only solution is $x = \dfrac{1}{4}$.

b. $\tan^{-1}$ is increasing, with values between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ (its graph rises from one asymptote to the other). So the left side, a sum of two increasing functions, is increasing: it rises from near $-\pi$ (both terms near $-\dfrac{\pi}{2}$ for large negative x) to near π, taking every value in between exactly once. So the equation has **exactly one solution for each c with $-\pi < c < \pi$**, and none otherwise. Taking tangents can't tell c from $c - \pi$ (or $c + \pi$): the quadratic of a. holds the root for $c = \tan^{-1}\dfrac{8}{31}$ and the root for $c = \tan^{-1}\dfrac{8}{31} - \pi$, which is also inside $(-\pi, \pi)$, one each. So one of the two had to go. (The tempting range $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is the range of one $\tan^{-1}$, not of the sum.)

MARKING SCHEME

- ✓ The tangent of both sides: $\dfrac{2x}{2-x^2} = \dfrac{8}{31}$.
- ✓ $x = \dfrac{1}{4}$ or -8.
- ✓ $x = \dfrac{1}{4}$ checked (product less than 1).
- ✓ $x = -8$ rejected: the sum is $\tan^{-1}\dfrac{8}{31} - \pi$.
- ✓ b. The left side increasing from $-\pi$ to π: one solution for each c in $(-\pi, \pi)$.
- ✓ So a. has one root; the other root of the quadratic belongs to $c = \tan^{-1}\dfrac{8}{31} - \pi$.

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking

4 Three inverse cosines

ANSWER HA · 5

Tests: moving one term across, cos $(A + B)$ with $\sqrt{1-x^2}$ for the sines, then squaring to clear the roots; squaring can't be undone.

a.

Let $A = \cos^{-1} x$, $B = \cos^{-1} y$: their sines are $\sqrt{1-x^2}$, $\sqrt{1-y^2}$ (not negative, as A, B are in $[0, \pi]$). $A + B = \pi - \cos^{-1} z$, so $\cos(A + B) = -z$: $xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$. Then $xy + z = \sqrt{(1-x^2)(1-y^2)}$; squaring, $x^2y^2 + 2xyz + z^2 = 1 - x^2 - y^2 + x^2y^2$, so $x^2 + y^2 + z^2 + 2xyz = 1$. ✓

Checks: $\frac{1}{2}$: each angle $\frac{\pi}{3}$, sum π; $\frac{3}{4} + \frac{2}{8} = 1$. ✓ $x = y = 0$: $\frac{\pi}{2} + \frac{\pi}{2} + \cos^{-1} z = \pi$ gives $z = 1$, and $0 + 0 + 1 + 0 = 1$. ✓

b. He is wrong. With $x = y = \frac{1}{2}$ the relation is $\frac{1}{2} + z^2 + \frac{z}{2} = 1$, that is $2z^2 + z - 1 = (2z - 1)(z + 1) = 0$: $z = \frac{1}{2}$ or $z = -1$. For $z = -1$ the sum is $\frac{\pi}{3} + \frac{\pi}{3} + \pi = \frac{5\pi}{3}$, not π, though $\frac{1}{4} + \frac{1}{4} + 1 - \frac{1}{2} = 1$. The step that can't be turned round is the squaring: $(xy + z)^2 = (1 - x^2)(1 - y^2)$ also holds when $xy + z$ is the negative square root, $-\sqrt{(1-x^2)(1-y^2)}$: here $xy + z = \frac{1}{4} - 1 = -\frac{3}{4}$. Then $\cos(A + B) = xy - \frac{3}{4} = -\frac{1}{2}$, not $-z = 1$, so $A + B \neq \pi - \cos^{-1} z$.

MARKING SCHEME

- ✓ $\cos(A + B) = -z$ from $A + B = \pi - \cos^{-1} z$.
- ✓ $xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$.
- ✓ Squared and simplified to $x^2 + y^2 + z^2 + 2xyz = 1$.
- ✓ Both checks.
- ✓ b. $z = -1$ also fits, with the sum $\frac{5\pi}{3}$: the converse fails at the squaring step.

5 Designing a formula for π

ANSWER HA · 4

Tests: the difference rule as a design tool; judging which argument limits a series.

a. $x = \tan\left(\tan^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{5}\right) = \dfrac{\frac{1}{3} - \frac{1}{5}}{1 + \frac{1}{15}} = \dfrac{2/15}{16/15} = \frac{1}{8}$ (the difference rule, $\frac{1}{15} > -1$). Put it into the known line: $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$. ✓

b. No: it is a little slower, if anything. In $\tan^{-1} x = x - \frac{x^3}{3} + \cdots$ each term is about x^2 times the one before. Both formulas still contain $\tan^{-1}\frac{1}{2}$, whose terms shrink only by about $\frac{1}{4}$ each, so it needs the most terms, and it is the same series in both. Splitting $\tan^{-1}\frac{1}{3}$ into two smaller pieces doesn't help the slowest series; the new formula only adds a third series to work out (for 10 decimal places, about 15 terms at $\frac{1}{2}$ either way, then about 9 at $\frac{1}{3}$ against 7 at $\frac{1}{5}$ and 5 at $\frac{1}{8}$: 24 terms against 27). Machin's formula is fast because it has no fraction larger than $\frac{1}{5}$.

MARKING SCHEME

- ✓ a. $x = \frac{1}{8}$ by the difference rule.
- ✓ The three-term formula from the known line.
- ✓ b. Each term is about x^2 times the last: the largest fraction limits the speed.
- ✓ Both keep $\frac{1}{2}$, so the new one is not faster in that way.

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking

6 The best place to stand

ANSWER HA · 5

Tests: a difference of inverse tangents in a real setting; with Unit 7's discriminant, deciding what is possible; the largest value.
(The sizes are made up.)

a. The top is 4 m above eye level and the bottom 1 m, so the lines to them make angles $\tan^{-1}\frac{4}{x}$ and $\tan^{-1}\frac{1}{x}$ with the level. By the difference rule ($\frac{4}{x^2} > -1$): $\tan^{-1}\dfrac{\frac{4}{x}-\frac{1}{x}}{1+\frac{4}{x^2}} = \tan^{-1}\dfrac{3x}{x^2+4}$.

b. $45°$ needs $\dfrac{3x}{x^2+4} = 1$: $x^2 - 3x + 4 = 0$, $D = 9 - 16 = -7 < 0$. No real x: never $45°$.

c. θ is largest when $\dfrac{3x}{x^2+4}$ is, that is when $x + \dfrac{4}{x}$ is least. $x + \dfrac{4}{x} - 4 = \dfrac{(x-2)^2}{x} \geq 0$, so the least value is 4, at $x = 2$. There $\tan\theta = \dfrac{3}{4}$, $\theta = \tan^{-1}\dfrac{3}{4} \approx 36.9°$. (Closer or farther, the angle is smaller: at 1 m about $31.0°$, at 4 m about $31.0°$.)

MARKING SCHEME

✓ a. The two angles $\tan^{-1}\frac{4}{x}$, $\tan^{-1}\frac{1}{x}$.

✓ The difference rule gives $\tan^{-1}\dfrac{3x}{x^2+4}$.

✓ b. $x^2 - 3x + 4 = 0$ has $D < 0$: never $45°$.

✓ c. $x + \dfrac{4}{x} \geq 4$, least at $x = 2$.

✓ $\theta = \tan^{-1}\dfrac{3}{4} \approx 36.9°$.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	0	29	**29**

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking