Unit 9 · Inverse Circular Functions

ANSWER KEY · FOR TEACHERS

Full marks 40

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 4

1. one-to-one
2. principal value
3. branch
4. domain
5. range
6. mirror image
7. asymptotes
8. arcsin

Each carries 0.5 mark.

2 True or false

U · 3

i. ✗ **False**: $\frac{1}{\sin x}$ is cosec x; $\sin^{-1} x$ is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x.

ii. ✓ **True**

iii. ✗ **False**: Only for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$: $\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \frac{\pi}{6}$.

iv. ✓ **True**

v. ✗ **False**: Only when $xy < 1$: $\tan^{-1} 2 + \tan^{-1} 3 = \frac{3\pi}{4}$, not $\tan^{-1}(-1) = -\frac{\pi}{4}$.

vi. ✓ **True**

Each carries 0.5 mark.

3 Multiple choice

i. **c.** $[-1, 1]$ — K · 1

$\cos^{-1} x$ takes a cosine and gives an angle, and a cosine is between -1 and 1. $[0, \pi]$ is its range.

ii. **a.** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ — K · 1

The open interval: tan is undefined at $\pm\frac{\pi}{2}$, so $\tan^{-1} x$ never reaches them. $(0, \pi)$ is $\cot^{-1}$'s range.

iii. **d.** $\frac{3\pi}{4}$ — U · 1

$\cot^{-1}(-x) = \pi - \cot^{-1} x = \pi - \frac{\pi}{4}$. $-\frac{\pi}{4}$ is outside $(0, \pi)$; $\frac{5\pi}{4}$ has cotangent $+1$.

iv. **b.** $-\frac{5}{12}$ — U · 1

The angle is in $\left(-\frac{\pi}{2}, 0\right)$, with sine $-\frac{5}{13}$ and cosine $+\frac{12}{13}$: tangent $-\frac{5}{12}$. $-\frac{12}{5}$ is the cotangent.

v. **c.** $-\frac{1}{2}$ — A · 1

$\frac{x+3}{1-3x} = 1$ gives $x = -\frac{1}{2}$; then $3x = -\frac{3}{2} < 1$, so the rule applies.

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vi. **a.** $3\pi - 10$ HA · 1

The answer must be in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with the same sine as 10. $\sin(3\pi - \theta) = \sin\theta$, and $3\pi - 10 \approx -0.575$ is in the branch. $10 - 3\pi$ is in the branch too but has the opposite sine.

4 Short answers

a. MARKING SCHEME U · 2

✓ $\frac{3\pi}{4}$.

✓ $-\frac{\pi}{3}$.

b. MARKING SCHEME U · 2

✓ $\frac{7\pi}{6}$ is outside $[0, \pi]$.

✓ $\frac{5\pi}{6}$.

c. MARKING SCHEME A · 2

✓ $\sqrt{1 - \frac{25}{169}} = \frac{12}{13}$.

✓ Positive: the angle is in $[0, \pi]$ (second quadrant).

d. MARKING SCHEME A · 2

✓ $x = -\frac{1}{2}$.

✓ $\sin^{-1} x = -\frac{\pi}{6}$, checked.

e. MARKING SCHEME A · 3

✓ $\frac{\pi}{4} = \tan^{-1} 1$, and the difference rule.

✓ Its condition $x > -1$ named.

✓ b. $-\frac{\pi}{12}$.

5 Structured question

ANSWER A · 5

a. $\dfrac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{(x-1)(x+1)}{(x-2)(x+2)}} = 1$. Times $(x-2)(x+2)$: top $2x^2 - 4$, bottom $(x^2 - 4) - (x^2 - 1) = -3$. So $\frac{2x^2 - 4}{-3} = 1$, $2x^2 = 1$, $x = \pm\frac{1}{\sqrt{2}}$.

b. For both, the product of the two values is $\frac{x^2 - 1}{x^2 - 4} = \frac{1}{7} < 1$, so the sum rule holds and both satisfy the equation. A check is needed because different angles share a tangent: taking tangents can add roots.

MARKING SCHEME

✓ The sum rule written for the two values.

✓ Cleared: $\frac{2x^2 - 4}{-3} = 1$.

✓ $x = \pm\frac{1}{\sqrt{2}}$.

✓ Both checked (product $\frac{1}{7} < 1$).

✓ Taking tangents can add roots.

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6 Long question

ANSWER A · 8

a. Let $\theta = \tan^{-1} x$, $x \geq 0$: then $0 \leq \theta < \frac{\pi}{2}$ and $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - x^2}{1 + x^2}$. Since $0 \leq 2\theta < \pi$ (here $x \geq 0$ is used), 2θ is in $\cos^{-1}$'s branch, so $\cos^{-1} \frac{1 - x^2}{1 + x^2} = 2\theta$.

b. $\sin^{-1}(1 - x) = \cos^{-1} x - \sin^{-1} x$. Sine of both sides: $1 - x = \sqrt{1 - x^2}\sqrt{1 - x^2} - x \cdot x = 1 - 2x^2$, so $2x^2 - x = 0$: $x = 0$ or $\frac{1}{2}$. Check: $0 + \frac{\pi}{2} = \cos^{-1} 0$ ✓; $\frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3} = \cos^{-1} \frac{1}{2}$ ✓.

c. $2 \times 3 = 6 > 1$ with both positive, so $\tan^{-1} 2 + \tan^{-1} 3 = \pi + \tan^{-1} \frac{5}{1 - 6} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. Adding $\tan^{-1} 1 = \frac{\pi}{4}$ gives π.

MARKING SCHEME

- ✓ a. $\cos 2\theta = \frac{1 - x^2}{1 + x^2}$.
- ✓ 2θ in $[0, \pi)$ because $x \geq 0$.
- ✓ So $\cos^{-1} \frac{1 - x^2}{1 + x^2} = 2 \tan^{-1} x$.
- ✓ b. Rearranged and the sine of both sides taken.
- ✓ $1 - x = 1 - 2x^2$: $x = 0$ or $\frac{1}{2}$.
- ✓ Both checked.
- ✓ c. $xy > 1$, so π added: $\frac{3\pi}{4}$.
- ✓ Plus $\frac{\pi}{4}$: π.

7 Think

ANSWER HA · 3

The identity holds only for $|x| \leq 1$. With $\theta = \tan^{-1} 2$, $2\theta \approx 2.214$ is more than $\frac{\pi}{2}$, outside $\sin^{-1}$'s branch; $\sin 2\theta = \frac{4}{5}$ still, but $\sin^{-1}$ gives the angle in the branch with that sine, $\pi - 2\theta$. So $2.214 + 0.927 = \pi$.

MARKING SCHEME

- ✓ The identity needs $|x| \leq 1$.
- ✓ 2θ is outside $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}$ gives $\pi - 2\theta$.
- ✓ The two add up to π.

8 Activity: One principal value each

(not marked)

WHAT STUDENTS SHOULD FIND

$\sin^{-1} x$: 30°, 53.1°, −30°, 90°, −90°; $\cos^{-1} x$: 60°, 36.9°, 120°, 0°, 180°; every sum 90°. Other angles: 150°, 126.9°, 210°, 450°, 270° (or any of the many others). The calculator gives only the principal value, the one angle in the branch.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	6	9	21	4	**40**

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