Unit 7 · Quadratic Equation

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 26

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Where the roots lie

ANSWER HA · 6

Tests: the nature of the roots needs the discriminant, the sum and the product together; the tempting check of the sum alone.

a. $\frac{D}{4} = k^2 - (2k + 3) = (k - 3)(k + 1) \geq 0$: $k \leq -1$ or $k \geq 3$.

b. Sum $2k$, product $2k + 3$. Both positive: real, sum > 0, product > 0: $k \geq 3$. Both negative: real, sum < 0, product > 0: $k < 0$ and $k > -\frac{3}{2}$, with $k \leq -1$: $-\frac{3}{2} < k \leq -1$. Opposite signs: product < 0, $k < -\frac{3}{2}$ (then the roots are real by themselves, since $k < -1$).

c. No. At $k = -2$ the product is $-1 < 0$, so the roots have opposite signs: $x^2 + 4x - 1 = 0$, $x = -2 \pm \sqrt{5}$, about 0.24 and -4.24. The negative sum says only that the negative root is the larger in size.

MARKING SCHEME

✓ a. $\frac{D}{4} = (k - 3)(k + 1) \geq 0$: $k \leq -1$ or $k \geq 3$.

✓ b. Both positive: $k \geq 3$.

✓ Both negative: $-\frac{3}{2} < k \leq -1$.

✓ Opposite signs: $k < -\frac{3}{2}$.

✓ c. No: the product is negative, so the signs differ.

✓ The roots $-2 \pm \sqrt{5}$.

2 Never real, except once

ANSWER A · 4

Tests: the discriminant rearranged into a square; the boundary case.

a. $\frac{D}{4} = (ac + bd)^2 - (a^2 + b^2)(c^2 + d^2) = a^2c^2 + 2abcd + b^2d^2 - a^2c^2 - a^2d^2 - b^2c^2 - b^2d^2 = -(ad - bc)^2$. This is negative unless $ad = bc$: no real roots.

b. If $ad = bc$, $D = 0$: equal roots, $x = -\frac{ac + bd}{a^2 + b^2}$. With $d = \frac{bc}{a}$: $ac + bd = \frac{c(a^2 + b^2)}{a}$, so $x = -\frac{c}{a}$.

Check: $5x^2 + 30x + 45 = 5(x + 3)^2$, root $-3 = -\frac{3}{1}$. ✓

MARKING SCHEME

✓ a. $\frac{D}{4} = (ac + bd)^2 - (a^2 + b^2)(c^2 + d^2)$.

✓ Expanded to $-(ad - bc)^2$, negative unless $ad = bc$.

✓ b. Equal roots $-\frac{ac + bd}{a^2 + b^2} = -\frac{c}{a}$.

✓ The check: $5(x + 3)^2$, root -3.

3 New roots from old

ANSWER

Tests: building the new sum and product from $\alpha + \beta$ and $\alpha\beta$; multiplying out fully; two equations are the same when their sums and products are.

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a. $\alpha+\beta=\frac{5}{2}$, $\alpha\beta=\frac{1}{2}$. New sum: $(\alpha+\beta)+\frac{\alpha+\beta}{\alpha\beta}=\frac{5}{2}+5=\frac{15}{2}$. New product: $\left(\alpha+\frac{1}{\beta}\right)\left(\beta+\frac{1}{\alpha}\right)=\alpha\beta+1+1+\frac{1}{\alpha\beta}=\frac{1}{2}+2+2=\frac{9}{2}$. So $x^2-\frac{15}{2}x+\frac{9}{2}=0$, that is $2x^2-15x+9=0$.

b. The two cross terms $\alpha\cdot\frac{1}{\alpha}=1$ and $\frac{1}{\beta}\cdot\beta=1$: her $\frac{5}{2}$ is 2 short.

c. With $\alpha+\beta=-p$, $\alpha\beta=q$: the new sum is $-p-\frac{p}{q}$ and the new product $q+2+\frac{1}{q}$. The two pairs are the same exactly when the sums and the products agree. Sums: $\frac{p}{q}=0$, so $p=0$. Products: $2+\frac{1}{q}=0$, so $q=-\frac{1}{2}$. Just one equation: $x^2-\frac{1}{2}=0$. Check: its roots are $\pm\frac{1}{\sqrt{2}}$, and $\frac{1}{\sqrt{2}}+\frac{1}{-1/\sqrt{2}}=\frac{1}{\sqrt{2}}-\sqrt{2}=-\frac{1}{\sqrt{2}}$: each new root is the other old one. ✓ (Trying $\alpha+\frac{1}{\beta}=\alpha$ directly is a dead end, since $\frac{1}{\beta}$ is never 0; comparing sum and product finds the swapped order without guessing it.)

MARKING SCHEME

✓ a. New sum $\frac{15}{2}$.

✓ New product $\frac{9}{2}$, with the cross terms.

✓ $2x^2-15x+9=0$.

✓ b. The cross terms $1+1$ left out.

✓ c. Equal sums and products: $p=0$, $q=-\frac{1}{2}$.

✓ The only equation, $x^2-\frac{1}{2}=0$, checked.

4 After the common root

ANSWER HA · 5

Tests: the common root by subtracting; then using sum or product to reach the roots not shared, and their signs from their sum and product.

a. Subtracting: $(a-b)x+(b-a)=0$, and $a\neq b$, so $x=1$. Put in the first: $1+a+b=0$.

b. The first equation's roots multiply to b, and one is 1, so the other is b; likewise the second's other root is a. An equation with roots a and b: $x^2-(a+b)x+ab=0$, with $a+b=-1$ and $b=-1-a$: $x^2+x-a(a+1)=0$.

c. Their sum is $a+b=-1<0$, so they are **never both positive**. Both negative needs a positive product as well, $-a(a+1)>0$: $-1<a<0$, leaving out $a=-\frac{1}{2}$, where $b=a$ (for example $a=-\frac{1}{4}$, $b=-\frac{3}{4}$). For $a>0$ or $a<-1$ one is positive and one negative (at $a=0$ or -1 one of them is 0). Check, $a=2$, $b=-3$: $x^2+2x-3=(x-1)(x+3)$, $x^2-3x+2=(x-1)(x-2)$: common root 1, others -3 and 2, of opposite signs, and $x^2+x-6=(x+3)(x-2)$. ✓

MARKING SCHEME

✓ a. Subtracting gives $x=1$ (using $a\neq b$).

✓ $a+b+1=0$.

✓ b. The other roots b and a, from the products.

✓ Their equation $x^2+x-a(a+1)=0$.

✓ c. Never both positive (sum -1); both negative for $-1<a<0$, $a\neq-\frac{1}{2}$; the check with $a=2$.

5 A garden against a wall

ANSWER

Tests: setting up the quadratic in a real setting; the discriminant as "possible or not"; the largest value where $D=0$; the tempting square.

a. $A=x(30-2x)$, so $2x^2-30x+A=0$, with $D=900-8A$.

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b. 100: $D = 100 > 0$, $x = 5$ or 10: 5 m by 20 m or 10 m by 10 m. 112: $D = 4 > 0$, $x = 7$ or 8: 7 m by 16 m, or 8 m by 14 m. 120: $D = -60 < 0$: impossible.

c. The largest A has $D = 0$: $A = \frac{900}{8} = 112.5$ m^2, at $x = \frac{30}{4} = 7.5$: 7.5 m out from the wall and 15 m along it. Not a square: the side along the wall is twice the others, because the wall gives that side free. (The 10 m by 10 m square encloses only 100 m^2.)

MARKING SCHEME

- ✓ a. $2x^2 - 30x + A = 0$, $D = 900 - 8A$.
- ✓ b. 100 m^2 and 112 m^2 possible, with their sides.
- ✓ 120 m^2 impossible ($D < 0$).
- ✓ c. $D = 0$: 112.5 m^2, 7.5 m by 15 m.
- ✓ Not a square: the side along the wall is twice the other.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	9	17	**26**

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