Unit 6 · Matrices and Determinants

HARDER PRACTICE QUESTIONS

Full marks 28

Write your answers on separate paper. Show your working. Name the property you use at each step.

1 Two symmetric matrices [5]

A and B are symmetric matrices of the same order.

a. Prove that AB is symmetric if and only if $AB = BA$.

b. Prove that $AB - BA$ is always skew-symmetric.

c. $P = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. Find every symmetric 2×2 matrix B for which PB is symmetric, and show that each is of the form $pI + qP$ for numbers p, q.

2 A chain of determinants [5]

A is a 3×3 matrix with $|A| = -2$.

a. Find $|\text{adj } A|$, $|2A^{-1}|$ and $|\text{adj}(\text{adj } A)|$.

b. Bibek knows only that a 3×3 matrix B has $|\text{adj } B| = 16$. He writes "$|B| = 4$". What can he really say?

c. Using $A^{-1} = \dfrac{1}{|A|}\text{adj } A$ twice, show that $\text{adj}(\text{adj } A) = |A|A$ for any 3×3 matrix with $|A| \neq 0$. Check your answer to a. with it.

3 When is it singular? [4]

a. Show, by the properties of determinants, that $\det \begin{pmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{pmatrix} = abc + ab + bc + ca$.

b. Show that the matrix has an inverse whenever a, b, c are all positive. Ramesh says the same is true when they are all negative. Find negative values that prove him wrong, and give the condition, for a, b, c all non-zero, for the matrix to have no inverse.

4 A plot with a corner pointing in [5]

A surveyor writes the corners of a plot, in meters, in order round it: $P(0, 0)$, $Q(20, 0)$, $R(20, 20)$, $S(10, 8)$, $T(0, 20)$. The corner S points into the plot.

a. Nischal cuts the plot into triangles from P: PQR, PRS, PST. He finds each area as half the size of its determinant and adds them. Find his total.

b. Find the true area by cutting from S instead, and check it another way.

c. One of Nischal's determinants is negative. Show that if he had kept the signs (adding half of each determinant as it is), he would have got the true area. Why is that determinant negative, and why must its triangle be taken away rather than added?

5 An inverse without the adjoint [4]

a. A square matrix A satisfies $A^2 = A + 2I$. Show that A has an inverse, and write A^{-1} in terms of A and I.

b. Check a. with $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$.

c. A square matrix B satisfies $B^2 = B$. Shreya says B can have an inverse only if $B = I$. Prove that she is right, and show that $|B|$ can only be 0 or 1.

6 A code in threes [5]

Letters are numbers (A = 1, …, Z = 26, a space 0), sent in blocks of three as $K\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with the key $K = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

a. Find K^{-1}, and decode the message 26, 22, 39, 25, 12, 25.

b. Pasang suggests the key $L = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{pmatrix}$. Show that the blocks EFA and DEB are sent as the same three numbers.

c. Explain, using determinants, why no key of this kind (with $|L| = 0$) can be used, whatever the message.

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