Unit 6 · Matrices and Determinants

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 28

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Two symmetric matrices

ANSWER HA · 5

Tests: the reversal law; matrix products don't commute, so "symmetric times symmetric" need not be symmetric.

a. $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}} = BA$ (reversal law, then A, B symmetric). So $(AB)^{\mathrm{T}} = AB$ exactly when $BA = AB$.

b. $(AB - BA)^{\mathrm{T}} = (AB)^{\mathrm{T}} - (BA)^{\mathrm{T}} = BA - AB = -(AB - BA)$: skew-symmetric. ✓

c. Let $B = \begin{pmatrix} p & q \\ q & r \end{pmatrix}$. By a., PB is symmetric exactly when $PB = BP$; $PB = \begin{pmatrix} p+2q & q+2r \\ 2p+3q & 2q+3r \end{pmatrix}$, symmetric when $q + 2r = 2p + 3q$, that is $r = p + q$. So $B = \begin{pmatrix} p & q \\ q & p+q \end{pmatrix} = pI + q\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}(P - I)$: $B = \left(p - \frac{q}{2}\right)I + \frac{q}{2}P$, of the form asked. (So most symmetric B fail: $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ gives $PB = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$, not symmetric.) The tempting answer "a product of symmetric matrices is symmetric" would need $(AB)^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$, the order kept.

MARKING SCHEME

✓ a. $(AB)^{\mathrm{T}} = BA$ by the reversal law.

✓ So symmetric if and only if $AB = BA$.

✓ b. $(AB - BA)^{\mathrm{T}} = -(AB - BA)$.

✓ c. PB symmetric exactly when $r = p + q$.

✓ Written as $pI + qP$ with $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}(P - I)$.

2 A chain of determinants

ANSWER A · 5

Tests: the rules $|kA| = k^3|A|$, $|\operatorname{adj} A| = |A|^2$, $|A^{-1}| = \dfrac{1}{|A|}$ combined; the lost negative root.

a. $|\operatorname{adj} A| = (-2)^2 = 4$. $|2A^{-1}| = 2^3 \times \dfrac{1}{-2} = -4$ (the tempting $\dfrac{2}{-2} = -1$ doubles only one row). $|\operatorname{adj}(\operatorname{adj} A)| = |\operatorname{adj} A|^2 = 16$.

b. $|B|^2 = 16$, so $|B| = 4$ or $|B| = -4$: the adjoint's determinant can't tell them apart (A of a. has $|A| = -2$ and $|\operatorname{adj} A| = 4$, just as a matrix with determinant 2 would).

c. $\operatorname{adj} A = |A|A^{-1}$, and it is non-singular ($|\operatorname{adj} A| = |A|^2 \neq 0$). Applying the same rule to it: $\operatorname{adj}(\operatorname{adj} A) = |\operatorname{adj} A|(\operatorname{adj} A)^{-1} = |A|^2 \cdot \dfrac{1}{|A|}(A^{-1})^{-1} = |A|A$. Check: $|(-2)A| = (-2)^3(-2) = 16$. ✓

MARKING SCHEME

✓ a. $|\operatorname{adj} A| = 4$.

✓ $|2A^{-1}| = -4$ and $|\operatorname{adj}(\operatorname{adj} A)| = 16$.

✓ b. $|B| = \pm 4$: he can't decide the sign.

✓ c. $\operatorname{adj} A = |A|A^{-1}$, and $(\operatorname{adj} A)^{-1} = \dfrac{1}{|A|}A$.

✓ $\operatorname{adj}(\operatorname{adj} A) = |A|A$, and the check 16.

3 When is it singular?

ANSWER

Tests: choosing the row operations that make zeros; reading the result to find the singular cases, beyond trying numbers.

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a. $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ (adding a multiple of a row changes nothing): rows $(1+a, 1, 1)$, $(-a, b, 0)$, $(-a, 0, c)$. Expand along column 3:
$1 \times \det\begin{pmatrix} -a & b \\ -a & 0 \end{pmatrix} - 0 + c \times \det\begin{pmatrix} 1+a & 1 \\ -a & b \end{pmatrix} = ab + c(b + ab + a) = abc + ab + bc + ca$. ✓

b. If $a, b, c > 0$, every term of $abc + ab + bc + ca$ is positive, so the determinant is not 0 and the inverse exists. For negative values the terms have mixed signs ($abc < 0$, the rest > 0), so they can cancel: $a = b = c = -3$ gives $-27 + 27 = 0$, a singular matrix (each row then adds up to 0, so $C_1 \to C_1 + C_2 + C_3$ makes a column of zeros). For a, b, c all non-zero, take out abc: the determinant is $abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$, so the matrix has no inverse exactly when $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = -1$ (for example $-2, -4, -4$ too). (The tempting argument "all the same sign, so no cancelling" forgets that abc has the sign of a, and ab doesn't.)

MARKING SCHEME

- ✓ a. Row operations that make zeros (named).
- ✓ The expansion carried to $abc + ab + bc + ca$.
- ✓ b. All terms positive for positive a, b, c; a negative example such as $-3, -3, -3$.
- ✓ The condition $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = -1$.

4 A plot with a corner pointing in

ANSWER HA · 5

Tests: the triangle-area determinant and its sign; the fan of triangles fails when a triangle lies outside the plot.

a. $\det\begin{pmatrix} 0 & 0 & 1 \\ 20 & 0 & 1 \\ 20 & 20 & 1 \end{pmatrix} = 400$, $\det\begin{pmatrix} 0 & 0 & 1 \\ 20 & 20 & 1 \\ 10 & 8 & 1 \end{pmatrix} = -40$, $\det\begin{pmatrix} 0 & 0 & 1 \\ 10 & 8 & 1 \\ 0 & 20 & 1 \end{pmatrix} = 200$. His total: $\frac{1}{2}(400 + 40 + 200) = 320\ \text{m}^2$.

b. From S: SPQ, SQR, STP all lie inside the plot, with areas 80, 100 and 100 m^2 (base 20, heights 8, 10, 10, or by determinants 160, 200, 200 halved): **280 m^2**. Check: the square $PQRT$ is 400 m^2, less the notch RST, $\frac{1}{2} \times 20 \times 12 = 120$: 280 m^2. ✓ (About 8.8 aana, at 31.80 m^2 an aana.)

c. $\frac{1}{2}(400 - 40 + 200) = 280$. ✓ Going $P \to R \to S$ turns clockwise, the other way from $P \to Q \to R$ and $P \to S \to T$, so its determinant is negative. The triangles from P overlap: a point of the notch such as (15, 14.5) lies in PQR and in PRS, counted $+1 - 1 = 0$ times; a point of the plot such as (3, 2.9) lies in PQR, PRS and PST, counted $1 - 1 + 1 = 1$ time. With the signs kept every point of the plot is counted once and the notch not at all. Taking every area as positive adds PRS instead of taking it away: $280 + 2 \times 20 = 320$, Nischal's answer.

MARKING SCHEME

- ✓ a. The three determinants 400, −40, 200.
- ✓ His total 320 m^2.
- ✓ b. 280 m^2 from S.
- ✓ The check by square minus notch.
- ✓ c. Signs kept give 280; PRS runs clockwise, and its area must be taken away where the triangles from P overlap.

5 An inverse without the adjoint

ANSWER

Tests: a matrix equation as a route to the inverse; what $|AB| = |A||B|$ says about such equations.

a. $A^2 - A = 2I$, so $A \cdot \frac{1}{2}(A - I) = I$ (and the same from the other side, since A commutes with $A - I$). So $A^{-1} = \frac{1}{2}(A - I)$.

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b. $A^2 = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} = A + 2I$. ✓ $\frac{1}{2}(A - I) = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$, and A times it is I. ✓ (Also $|A| = -2$, adj $A = \begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix}$, the same.) HA · 4

c. If B^{-1} exists, multiply $B^2 = B$ by it: $B = I$. And $|B|^2 = |B^2| = |B|$, so $|B|(|B| - 1) = 0$: $|B| = 0$ or 1. (A matrix like $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has $B^2 = B$ with $|B| = 0$: the tempting "$B^2 = B$ so $B = I$" divides by B without an inverse.)

MARKING SCHEME

✓ a. $A \cdot \frac{1}{2}(A - I) = I$, so $A^{-1} = \frac{1}{2}(A - I)$.
✓ b. The equation and the inverse checked.
✓ c. Multiplying by B^{-1} gives $B = I$.
✓ $|B|^2 = |B|$, so $|B| = 0$ or 1.

6 A code in threes

ANSWER HA · 5

Tests: the inverse as decoding; what a singular matrix does in use (two inputs, one output).

a. $|K| = 1(1 - 2) - 2(0 - 1) + 0 = 1$. Cofactors transposed: $K^{-1} = \begin{pmatrix} -1 & -2 & 2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$.
$K^{-1}(26, 22, 39) = (8, 9, 13)$: H, I, M; $K^{-1}(25, 12, 25) = (1, 12, 0)$: A, L, space. The message is **HIMAL**.

b. EFA is (5, 6, 1), DEB is (4, 5, 2): L sends both to (20, 40, 7). Their difference (1, 1, −1) is sent to (0, 0, 0).

c. Row 2 of L is twice row 1, so $|L| = 0$ and L has no inverse: there is no matrix that undoes it. Whenever $|L| = 0$, as in b., some block other than (0, 0, 0) is sent to zeros, so two different messages can be sent as the same numbers, and the receiver can't know which was meant.

MARKING SCHEME

✓ a. $|K| = 1$ and K^{-1} right.
✓ HIMAL decoded.
✓ b. Both blocks sent as (20, 40, 7).
✓ c. $|L| = 0$, so no inverse to decode with.
✓ Two messages give the same code, so decoding is impossible.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	9	19	**28**

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