Unit 6 · Matrices and Determinants

ANSWER KEY · FOR TEACHERS

Full marks 43

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 4

1. transpose
2. symmetric
3. skew-symmetric
4. minor
5. cofactor
6. adjoint
7. singular
8. triangular

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$: the order turns round (the reversal law).

iii. ✗ **False**: $|2A| = 2^3|A| = 8|A|$: all three rows are doubled.

iv. ✓ **True**

v. ✗ **False**: Not in general: for $A = B = I$ (3×3), $|A + B| = |2I| = 8$ but $|A| + |B| = 2$.

vi. ✓ **True**

Each carries 0.5 mark.

3 Multiple choice

i. **b.** the transpose of the matrix of cofactors of A — K · 1

The cofactor of a_{ij} goes to place (j, i): the matrix of cofactors, transposed. The first forgets to transpose; the last forgets the signs as well.

ii. **c.** -15 — U · 1

Strike out row 3 and column 2: $M_{32} = 3 \times 5 - (-2) \times 0 = 15$, and $C_{32} = (-1)^5(15) = -15$. 15 forgets the sign.

iii. **a.** $-3D$ — U · 1

The swap gives $-D$, and one row times 3 multiplies it by 3: $-3D$. $-27D$ multiplies every row by 3.

iv. **d.** 4 — A · 1

No inverse means a determinant of 0: $1(4k - 6) - 2(8 - 0) + 3(2 - 0) = 4k - 16 = 0$, so $k = 4$. Then row 2 is twice row 1.

v. **c.** 9 — A · 1

$A(\mathrm{adj}\, A) = |A|I$, so $|A||\mathrm{adj}\, A| = |A|^3$ and $|\mathrm{adj}\, A| = |A|^2 = 9$. $\frac{1}{3}$ is $|A^{-1}|$.

vi. **a.** 2 — HA · 1

$|A^3| = |A|^3$ and $|2I| = 2^3 = 8$, so $|A|^3 = 8$ and $|A| = 2$. 8 is $|2I|$, not yet cube-rooted; $\sqrt[3]{2}$ takes $|2I|$ to be 2.

4 Short answers

a. MARKING SCHEME — U · 2

✓ $AB = I$ gives $|A||B| = |I| = 1$.

✓ Impossible when $|A| = 0$, so there is no inverse.

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b. MARKING SCHEME U · 2

✓ The sign of place (3, 2) is − (cofactor, not minor).

✓ $|A| = 28$.

c. MARKING SCHEME A · 2

✓ $|B|^2 = 16$, so $|B| = 4$.

✓ $|2B^{-1}| = 8 \times \frac{1}{4} = 2$.

d. MARKING SCHEME A · 2

✓ The determinant with a column of 1s set to 0 (collinear).

✓ $4k - 4 = 0$, $k = 1$.

e. MARKING SCHEME A · 3

✓ S^{T} right.

✓ $P = \frac{1}{2}(S + S^{\mathrm{T}})$ right.

✓ $Q = \frac{1}{2}(S - S^{\mathrm{T}})$ right, zeros on its diagonal.

f. MARKING SCHEME A · 3

✓ $C_3 \to C_3 + C_2$, each entry $a + b + c$.

✓ $a + b + c$ taken out.

✓ Two equal columns, so 0.

5 Structured question

ANSWER A · 5

a. $A^2 = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$, $5A = \begin{pmatrix} 10 & -5 & 5 \\ -5 & 10 & -5 \\ 5 & -5 & 10 \end{pmatrix}$, so $A^2 - 5A + 4I$ has $6 - 10 + 4 = 0$ on the diagonal and 0 elsewhere: O. ✓

b. $|A| = 4 \neq 0$, so A^{-1} exists. Multiply by A^{-1}: $A - 5I + 4A^{-1} = O$, so

$$A^{-1} = \frac{1}{4}(5I - A) = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

MARKING SCHEME

✓ A^2 right.

✓ $5A$ and $4I$ subtracted and added.

✓ Every entry 0: O.

✓ Multiplied by A^{-1} (with $|A| \neq 0$): $A^{-1} = \frac{1}{4}(5I - A)$.

✓ The matrix $\frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$.

6 Long question

ANSWER

a. Cofactors: $C_{11} = 4$, $C_{12} = -5$, $C_{13} = 1$; $C_{21} = 2$, $C_{22} = 0$, $C_{23} = -2$; $C_{31} = 2$, $C_{32} = 5$, $C_{33} = 3$.

Transposed: $\operatorname{adj} A = \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}$.

b. $|A| = 1(4) + (-1)(-5) + 1(1) = 10$, so $A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}$. $A \times \operatorname{adj} A$ has rows (10, 0, 0), (0, 10, 0), (0, 0, 10), so $AA^{-1} = \frac{1}{10}(10I) = I$. ✓

c. $|A^{\mathrm{T}}| = |A| = 10$, and the cofactors of A^{T} are those of A, transposed, so $\operatorname{adj}(A^{\mathrm{T}}) = (\operatorname{adj} A)^{\mathrm{T}}$. Hence

$$(A^{\mathrm{T}})^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{pmatrix} = (A^{-1})^{\mathrm{T}}. \checkmark$$

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MARKING SCHEME A · 8

- ✓ a. Row 1's cofactors right.
- ✓ Rows 2 and 3's cofactors right.
- ✓ Transposed into the adjoint.
- ✓ b. $|A| = 10$.
- ✓ $A^{-1} = \frac{1}{10}\text{adj } A$.
- ✓ The check $AA^{-1} = I$ worked.
- ✓ c. $|A^{\mathrm{T}}| = 10$ and $\text{adj}(A^{\mathrm{T}})$ found.
- ✓ $(A^{\mathrm{T}})^{-1}$ equal to $(A^{-1})^{\mathrm{T}}$.

7 Think

ANSWER HA · 3

The cofactors of a matrix of whole numbers are 2×2 determinants of whole numbers, so adj A is whole; with $|A| = \pm 1$, $A^{-1} = \pm\text{adj } A$ is whole. Conversely, the determinant of a whole-number matrix is a whole number, so $|A|$ and $|A^{-1}|$ are whole numbers with $|A||A^{-1}| = |I| = 1$: two integers whose product is 1 are both 1 or both -1.

MARKING SCHEME

- ✓ adj A is whole, so $A^{-1} = \pm\text{adj } A$ is whole when $|A| = \pm 1$.
- ✓ $|A|$ and $|A^{-1}|$ are whole numbers with $|A||A^{-1}| = 1$.
- ✓ So $|A| = \pm 1$.

8 Activity: What each change does

(not marked)

WHAT STUDENTS SHOULD FIND

-15, -15 (transpose: the same), 15 (a swap changes the sign), -15 (adding a multiple of a row changes nothing), -30 (one row doubled: doubled), -120 (every row doubled: $2^3 = 8$ times).

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	5	9	25	4	**43**

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