Unit 5 · Sequence and Series

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 28

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 A sum with one term too many

ANSWER HA · 5

Tests: $t_n = S_n - S_{n-1}$ only for $n \geq 2$; the first term comes from S_1.

a. $t_1 = S_1 = 6$, $t_2 = S_2 - S_1 = 17 - 6 = 11$, $t_3 = 34 - 17 = 17$. For $n \geq 2$, $t_n = 6n - 1$, but $t_1 = 6$, not 5. The series is 6, 11, 17, 23, ...: the differences are 5, 6, 6, ..., so it is **not an AP**. She used the formula at $n = 1$, where S_0 has no meaning (her formula would need $S_0 = 1$, not 0).

b. For $n \geq 2$, $t_n = p(2n - 1) + q$, so $t_n - t_{n-1} = 2p$. And $t_1 = S_1 = p + q + c$, while the formula would give $p + q$. So $t_2 - t_1 = (3p + q) - (p + q + c) = 2p - c$, equal to $2p$ exactly when $c = 0$. Then the AP has **first term $p + q$ and common difference $2p$**.

MARKING SCHEME

- ✓ a. $t_1 = 6$, $t_2 = 11$, $t_3 = 17$.
- ✓ Not an AP: the first difference is 5, the others 6; t_1 must come from S_1.
- ✓ b. $t_n = p(2n - 1) + q$ for $n \geq 2$, difference $2p$.
- ✓ $t_2 - t_1 = 2p - c$: an AP exactly when $c = 0$.
- ✓ First term $p + q$, difference $2p$.

2 The three means at work

ANSWER A · 5

Tests: $G^2 = AH$ linking the three means; $A \geq G$ used to find a least or greatest value, and when equality holds.

a. $A = G + 2$, $H = G - \frac{8}{5}$, and $G^2 = AH = (G + 2)\left(G - \frac{8}{5}\right) = G^2 + \frac{2}{5}G - \frac{16}{5}$, so $G = 8$: **$A = 10$, $G = 8$, $H = 6.4$**. Then $a + b = 20$, $ab = 64$: the numbers are **16 and 4** (check: $H = \frac{2 \times 64}{20} = 6.4$ ✓).

b. With sides l and w, $lw = 400$ and the fence is $2(l + w) = 4 \cdot \frac{l + w}{2} \geq 4\sqrt{lw} = 80$ m, equal only when $l = w$: a **20 m by 20 m square, 80 m of fence**.

c. $l + w = 30$, so $\sqrt{lw} \leq \frac{l + w}{2} = 15$: area at most **225 m²**, a 15 m square.

MARKING SCHEME

- ✓ a. $G^2 = AH$ with $A = G + 2$, $H = G - \frac{8}{5}$: $G = 8$.
- ✓ The numbers 16 and 4.
- ✓ b. $2(l + w) \geq 4\sqrt{lw} = 80$ m.
- ✓ Least when $l = w = 20$ m.
- ✓ c. 225 m².

3 Paying the loan off sooner

ANSWER

Tests: the EMI as a GP sum; solving for the number of terms with a logarithm (Unit 3); interest depends on the balance over time.

a. $P = \frac{E}{1 + i} + \cdots + \frac{E}{(1 + i)^n} = \frac{E\left[1 - (1 + i)^{-n}\right]}{i}$ (a GP with first term $\frac{E}{1 + i}$, ratio $\frac{1}{1 + i}$), so $(1 + i)^{-n} = 1 - \frac{Pi}{E}$. ✓

b. $Pi = 16{,}666.67$, so $(1 + i)^{-n} = 1 - \frac{16{,}666.67}{25{,}000} = \frac{1}{3}$ and $(1 + i)^n = 3$: $n = \frac{\log 3}{\log (1.008333)} \approx 132.4$. So **132 full payments and a smaller 133rd** (about Rs 9,600): 11 years and 1 month instead of 15 years.

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c. Total paid about $132 \times 25{,}000 + 9{,}600 \approx$ Rs 33,10,000, against $180 \times 21{,}492 =$ Rs 38,68,560. The loan is the same Rs 20 lakh, so the difference is all interest: about **Rs 5.6 lakh saved** (interest about Rs 13.1 lakh instead of 18.7 lakh). The family pays less in all, not more: Rs 3,508 extra for 132 months (about Rs 4.6 lakh), then a smaller 133rd payment and nothing in the 47 months when the bank's plan still takes Rs 21,492 (about Rs 10.1 lakh). Abinash counts 180 months of extra, but the loan ends after 133, and each larger payment brings the balance, and so the next month's interest, down faster.

HA · 5

MARKING SCHEME

- ✓ a. The GP sum of the installments' values today, rearranged.
- ✓ b. $(1 + i)^n = 3$.
- ✓ $n \approx 132.4$: 133 payments, the last smaller.
- ✓ c. About Rs 5.6 lakh interest saved (from the totals).
- ✓ He is wrong: they pay less in all, since the loan ends after 133 months, not 180.

4 Three kinds in a row

ANSWER

HA · 4

Tests: the AP, GP and HP tests chained; choosing what to eliminate.

From the AP: $b = \frac{a+c}{2}$. From the GP: $c^2 = bd$, so $d = \frac{c^2}{b} = \frac{2c^2}{a+c}$. From the HP: $d = \frac{2ce}{c+e}$.

So $\frac{2c^2}{a+c} = \frac{2ce}{c+e}$, that is $c(c+e) = e(a+c)$, $c^2 + ce = ae + ce$, so $c^2 = ae$: a, c, e are in GP. ✓

MARKING SCHEME

- ✓ $b = \frac{a+c}{2}$ and $d = \frac{c^2}{b}$.
- ✓ $d = \frac{2ce}{c+e}$ from the HP.
- ✓ The two forms of d equated.
- ✓ Simplified to $c^2 = ae$.

5 A ball that bounces for ever

ANSWER

A · 5

Tests: an infinite GP where the quantity asked about (time) has a different ratio from the one given (height).

a. Down 10, then up and down 7.5, 5.625, ...:

$10 + 2(7.5 + 5.625 + \cdots) = 10 + 2 \cdot \frac{7.5}{1 - 3/4} = 10 + 60 =$ **70 m**.

b. The heights are $10\left(\frac{3}{4}\right)^k$, so the times are $\sqrt{\frac{2 \times 10}{g}}\left(\sqrt{\frac{3}{4}}\right)^k$: a GP with ratio $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \approx 0.866$.

c. $t_0 = \sqrt{\frac{20}{9.8}} \approx 1.429$ s. Total $= t_0 + 2(t_0 r + t_0 r^2 + \cdots) = t_0 \cdot \frac{1+r}{1-r} \approx 1.429 \times 13.93 \approx$ **19.9 s**. He used the heights' ratio for the times; time goes as the square root of height, so the ratio is $\sqrt{\frac{3}{4}}$, closer to 1, and the bounces take longer to die away.

MARKING SCHEME

- ✓ a. 70 m.
- ✓ b. Times in GP with ratio $\frac{\sqrt{3}}{2}$.
- ✓ c. $t_0 \approx 1.43$ s and the sum $t_0 \cdot \frac{1+r}{1-r}$.
- ✓ About 19.9 s.
- ✓ His mistake: time goes as $\sqrt{h}$, so the ratio is $\sqrt{3}/2$, not $\frac{3}{4}$.

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6 For which x does it add up?

ANSWER A · 4

Tests: convergence $|r| < 1$ with a variable ratio; an absolute value inequality read as distances.

a. It is a GP with ratio $r = \dfrac{x-1}{x+1}$, so it has a sum when $|x-1| < |x+1|$: x is closer to 1 than to -1, that is $x > \mathbf{0}$, $(0, \infty)$.

b. $S_\infty = \dfrac{1}{1 - \dfrac{x-1}{x+1}} = \dfrac{x+1}{2}$. $\dfrac{x+1}{2} = 3$ gives $x = 5$, which is in $(0, \infty)$ ✓ (ratio $\dfrac{2}{3}$).

MARKING SCHEME

✓ a. $\left|\dfrac{x-1}{x+1}\right| < 1$.

✓ $x > 0$, read as closer to 1 than to -1.

✓ b. $S_\infty = \dfrac{x+1}{2}$.

✓ $x = 5$, checked to be allowed.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	14	14	**28**

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