Unit 5 · Sequence and Series

ANSWER KEY · FOR TEACHERS

Full marks 42

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 4

1. sequence
2. series
3. common difference
4. common ratio
5. harmonic
6. geometric mean
7. arithmetic mean
8. converges

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: It is geometric: each term is 3 times the one before.

iii. ✗ **False**: The ratio is -1, and $|r| < 1$ fails: the partial sums swing between 1 and 0, so the series has no sum.

iv. ✓ **True**

v. ✗ **False**: Equal distances average harmonically: $\dfrac{2 \times 40 \times 60}{40 + 60} = 48$ km/h.

vi. ✗ **False**: That is the AP test; for an HP, $\dfrac{2}{b} = \dfrac{1}{a} + \dfrac{1}{c}$, so $b = \dfrac{2ac}{a + c}$.

Each carries 0.5 mark.

3 Multiple choice

i. **c.** $\dfrac{2ab}{a + b}$ — K · 1

The reciprocal of the average of the reciprocals: $\dfrac{1}{H} = \dfrac{1}{2}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$. The last is $\dfrac{1}{H}$, not H.

ii. **b.** -20 — U · 1

$a = 7$, $d = -3$: $t_{10} = 7 + 9(-3) = -20$. -23 uses $10d$ instead of $9d$; 20 loses the sign of d.

iii. **b.** $\sqrt{2}$ and $8\sqrt{2}$ — U · 1

$r = \dfrac{2}{\sqrt{2}} = \sqrt{2}$, and $t_7 = \sqrt{2}(\sqrt{2})^6 = \sqrt{2} \times 8 = 8\sqrt{2}$. 16 is the 8th term, $\sqrt{2}(\sqrt{2})^7$: one power too many.

iv. **b.** 12 — A · 1

$\dfrac{n}{2}[6 + 4(n - 1)] = 300$: $2n^2 + n - 300 = 0$, $(n - 12)(2n + 25) = 0$, so $n = 12$ (a number of terms can't be negative). Check: $6 \times (3 + 47) = 300$. ✓

v. **a.** 16 and 4 — A · 1

$a + b = 20$ and $ab = 64$: the roots of $x^2 - 20x + 64 = 0$, 16 and 4. 12 and 8 add to 20 but multiply to 96; 18 and 2 give G.M. 6.

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vi. **b.** $\frac{6}{5}$ HA · 1

$S = \frac{1}{1-2/3} = 3$. The even-placed terms are a GP with first term $\frac{2}{3}$ and ratio $\frac{4}{9}$: $E = \frac{2/3}{1-4/9} = \frac{6}{5}$. The odd-placed ones give $\frac{1}{1-4/9} = \frac{9}{5}$, and $\frac{6}{5} + \frac{9}{5} = 3$. ✓ $\frac{S}{2}$ would need the halves to be equal, but every even term is $\frac{2}{3}$ of the odd one before it.

4 Short answers

a. MARKING SCHEME U · 2

- ✓ The AP 2, 4, 6, 8, 10 of the reciprocals ($d = 2$).
- ✓ $\frac{1}{4}, \frac{1}{6}, \frac{1}{8}$.

b. MARKING SCHEME U · 3

- ✓ $r = \frac{3}{2}$: $|r| < 1$ fails, so the formula doesn't apply.
- ✓ The partial sums grow without end: no sum.
- ✓ Positive terms can't add up to a negative number.

c. MARKING SCHEME A · 3

- ✓ The repeating part as a GP with $a = \frac{34}{1000}$, $r = \frac{1}{100}$.
- ✓ Its sum $\frac{34}{990}$.
- ✓ $\frac{611}{495}$.

d. MARKING SCHEME U · 2

- ✓ A and H written and multiplied.
- ✓ $= ab = G^2$.

e. MARKING SCHEME A · 3

- ✓ $A = \frac{G^2}{H} = 7$.
- ✓ $a + b = 14$, $ab = 36$: $x^2 - 14x + 36 = 0$.
- ✓ $7 \pm \sqrt{13}$.

5 Structured question

ANSWER A · 5

a. $t_n = S_n - S_{n-1} = 3n^2 + 2n - 3(n-1)^2 - 2(n-1) = 6n - 1$ for $n \geq 2$, and $t_1 = S_1 = 5 = 6(1) - 1$ as well. $t_n - t_{n-1} = 6$, a constant: an AP with $a = 5$, $d = 6$.

b. $S_{20} - S_{10} = 1240 - 320 = \mathbf{920}$.

MARKING SCHEME

- ✓ a. $t_n = S_n - S_{n-1} = 6n - 1$.
- ✓ $t_1 = S_1 = 5$ fits the same formula.
- ✓ A constant difference 6: $a = 5$, $d = 6$.
- ✓ b. $S_{20} - S_{10}$ used.
- ✓ 920.

6 Long question

ANSWER

a. 500 mg; then $500 + \frac{1}{4}(500) = 625$ mg; then $500 + \frac{1}{4}(625) = 656.25$ mg.

b. Just after the nth dose the body holds that dose, a quarter of the one before, a sixteenth of the one before that, and so on back to the first:

$$500\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{n-1}}\right) = 500 \cdot \frac{1-(1/4)^n}{1-1/4} = \frac{2000}{3}\left(1 - \frac{1}{4^n}\right).$$

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c. As $n \to \infty$, $\frac{1}{4^n} \to 0$: it settles at $\frac{2000}{3} \approx$ **667 mg**, the sum of the infinite series. For a level L from a dose D, $L = \frac{D}{1 - 1/4} = \frac{4D}{3}$, so 800 mg needs $D =$ **600 mg**. A · 8

MARKING SCHEME

- ✓ a. 500 mg and 625 mg.
- ✓ 656.25 mg.
- ✓ b. The amount as a GP: $500\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{n-1}}\right)$.
- ✓ $a = 500$, $r = \frac{1}{4}$, n terms in the sum formula.
- ✓ Simplified to $\frac{2000}{3}\left(1 - \frac{1}{4^n}\right)$.
- ✓ c. $\frac{1}{4^n} \to 0$: the sum to infinity.
- ✓ About 667 mg.
- ✓ $\frac{4D}{3} = 800$: $D = 600$ mg.

7 Think

ANSWER HA · 3

The first month's interest is $20{,}00{,}000 \times \frac{0.10}{12} \approx$ Rs 16,667, so only about Rs 4,825 of the first installment repays the loan. In all the family pays $180 \times 21{,}492 =$ Rs 38,68,560, of which Rs 18,68,560 is interest. Interest is charged each month on what is still owed; an extra payment early cuts the balance on which every later month's interest is worked out, for all the years left.

MARKING SCHEME

- ✓ About Rs 16,667 interest, about Rs 4,825 of the loan.
- ✓ Total interest about Rs 18,68,560.
- ✓ Early payments reduce the balance on which all later interest is charged.

8 Activity: Recurring decimals as geometric series

(not marked)

WHAT STUDENTS SHOULD FIND

$0.\overline{7}$: $a = \frac{7}{10}$, $r = \frac{1}{10}$, $\frac{7}{9}$. $0.\overline{36}$: $a = \frac{36}{100}$, $r = \frac{1}{100}$, $\frac{4}{11}$. $0.4\overline{5}$: $\frac{4}{10}$ plus a GP with $a = \frac{5}{100}$, $r = \frac{1}{10}$: $\frac{41}{90}$. $0.\overline{142857}$: $a = \frac{142857}{10^6}$, $r = \frac{1}{10^6}$, $\frac{142857}{999999} = \frac{1}{7}$. A block of k digits gives $r = \frac{1}{10^k}$.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	5	12	21	4	**42**

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