Unit 4 · Curve Sketching

HARDER PRACTICE QUESTIONS

Full marks 28

Write your answers on separate paper. Show your working. Sketch on squared paper, marking intercepts, asymptotes and any point you use.

1 When does a sum repeat? [6]

a. Find the period of $g(x) = \sin\frac{x}{3} + \tan\frac{x}{2}$, and prove that no smaller positive number works.

b. Asmita says the period of $\cos^4 x - \sin^4 x$ is π, "the least common multiple of the periods of $\cos^4 x$ and $\sin^4 x$". Rohan says it is 2π. Who is right? Find the period.

c. The periods of $\sin x$ and $\sin(\sqrt{2}x)$ are 2π and $\sqrt{2}\pi$. Show that no positive number is a whole number of periods of both. What does that suggest about $\sin x + \sin(\sqrt{2}x)$?

2 Even parts and odd parts [5]

a. f is odd and g is even, both defined on all of $\mathbb{R}$. Decide whether each of $f \circ g$, $g \circ f$ and $f \circ f$ is even, odd or neither, with a proof.

b. Show that a function that is both even and odd is 0 everywhere on its domain.

c. Write $h(x) = \dfrac{1}{x-1}$ as an even function plus an odd function. On what domain can you do it, and why not on all of h's domain?

3 A hump found without calculus [5]

$f(x) = x(x-3)^2$.

a. Find the intercepts. Where does the curve cross the X-axis, and where does it only touch it?

b. Show that $4 - f(x) = (x-1)^2(4-x)$. Hence show that $f(x) \le 4$ for every $x \le 4$, with equality only at $x = 1$ and $x = 4$.

c. Sketch the curve.

d. For which values of k has $x(x-3)^2 = k$ three different real roots? How many has it for $k = 4$, for $k = 0$ and for $k = 5$?

4 A hyperbola moved [4]

a. Write $y = \dfrac{2x+1}{x-1}$ as $y = 2 + \dfrac{c}{x-1}$, and use it to give the asymptotes, the intercepts and where it increases or decreases. Sketch it.

b. Sneha solves $\dfrac{2x+1}{x-1} > 1$ by multiplying both sides by $x-1$: $2x+1 > x-1$, so $x > -2$. Show with one number that her answer is wrong, and find the right one from your sketch or algebra.

5 How fast do the days lengthen? [4]

Kathmandu's day length is about $L(t) = 12.14 + 1.76\sin\left(\dfrac{2\pi(t-80)}{365}\right)$ hours on day t of the year (the unit's model).

a. Find, in minutes, $L(81) - L(80)$ and $L(172) - L(171)$. Explain from the shape of a sine curve why one is so much bigger.

b. On which two days of the year is $L = 12.14$ hours? How far is each from the longest day, and why?

c. Use the model to count how many days of the year have more than 13 hours of daylight (a calculator or a table of values).

6 A line across the waves [4]

By sketching $y = \sin x$ and $y = \frac{x}{10}$ on the same axes, find how many real solutions $\sin x = \frac{x}{10}$ has. Explain why there are none for $|x| > 10$, why the count for $x < 0$ equals the count for $x > 0$, and which humps of the sine wave the line meets.