Unit 4 · Curve Sketching

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 28

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 When does a sum repeat?

ANSWER HA · 6

Tests: the LCM of the periods is a period but not always the least; a period must carry the points where a function is undefined onto each other; a sum need not repeat at all.

a. $\sin\frac{x}{3}$ has period 6π, $\tan\frac{x}{2}$ has period 2π; 6π is a whole number of both, so it is a period.

Nothing smaller: g is undefined exactly where $\tan\frac{x}{2}$ is, at $x = \pi + 2k\pi$, so a period T carries these points onto each other: T is a multiple of 2π. Then $\tan\frac{x}{2}$ repeats, so $\sin\frac{x+T}{3} = \sin\frac{x}{3}$ for all x, and T is a multiple of 6π. **Period 6π.**

b. Asmita's number is right, but not her reason.
$\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) = \cos 2x$: **period** π. $\cos^4 x$ and $\sin^4 x$ each have period π, and a sum of two functions of period π can repeat sooner (as $\sin^4 x + \cos^4 x$ does, with period $\frac{\pi}{2}$); here it doesn't, because the difference is $\cos 2x$. Rohan is wrong: 2π is a period but not the least.

c. A common multiple would need $2\pi m = \sqrt{2}\pi n$ for whole numbers $m, n \geq 1$, so $\sqrt{2} = \frac{2m}{n}$, a fraction: impossible, as $\sqrt{2}$ is irrational. So the two waves never come back into step together, which suggests that the sum never repeats. It is in fact **not periodic**; proving that takes more than counting periods, and isn't asked.

MARKING SCHEME

- ✓ a. 6π is a period (a whole number of each period).
- ✓ T a multiple of 2π, from where tan is undefined.
- ✓ Then a multiple of 6π: least period 6π.
- ✓ b. $\cos^4 x - \sin^4 x = \cos 2x$, period π; 2π not least.
- ✓ The LCM reason doesn't prove leastness.
- ✓ c. $\sqrt{2} = \frac{2m}{n}$ would be rational: no common multiple, so the sum is not expected to repeat.

2 Even parts and odd parts

ANSWER A · 5

Tests: the definitions used as tools, not only as tests; the even and odd parts need a domain symmetric about 0.

a. $f(g(-x)) = f(g(x))$: **even**. $g(f(-x)) = g(-f(x)) = g(f(x))$: **even**. $f(f(-x)) = f(-f(x)) = -f(f(x))$: **odd**.

b. $f(-x) = f(x)$ and $f(-x) = -f(x)$ give $f(x) = -f(x)$, so $f(x) = 0$.

c. The even part $\frac{h(x) + h(-x)}{2}$ needs $h(-x)$, so $x \neq \pm 1$. On $\mathbb{R} - \{-1, 1\}$: $h(x) = \frac{1}{x^2-1} + \frac{x}{x^2-1}$, even plus odd (the sum is $\frac{1+x}{x^2-1} = \frac{1}{x-1}$ ✓). Not on all of $\mathbb{R} - \{1\}$: $h(-1)$ exists but $h(1)$ doesn't, and an even or odd function's domain must contain $-x$ with x.

MARKING SCHEME

- ✓ a. $f \circ g$ even, $g \circ f$ even, with proofs.
- ✓ $f \circ f$ odd, with a proof.
- ✓ b. $f(x) = -f(x)$, so 0.
- ✓ c. $\frac{1}{x^2-1} + \frac{x}{x^2-1}$.
- ✓ On $\mathbb{R} - \{-1, 1\}$: the domain must be symmetric.

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3 A hump found without calculus

ANSWER HA · 5

Tests: a repeated factor touches; a factorization locates a hump exactly; counting the roots of $f(x) = k$ with a horizontal line.

a. $(0, 0)$: a single root, the curve crosses. $(3, 0)$: a double root, it touches (the sign doesn't change: $f \geq 0$ near 3). Y-intercept $(0, 0)$.

b. $(x-1)^2(4-x) = (x^2-2x+1)(4-x) = -x^3+6x^2-9x+4 = 4-(x^3-6x^2+9x) = 4-f(x)$. ✓ For $x \leq 4$, $(x-1)^2 \geq 0$ and $4-x \geq 0$, so $4-f(x) \geq 0$, zero only at $x = 1$ or $x = 4$.

c. Down on the left ($a = 1 > 0$), up through the origin to the hump $(1, 4)$, down to touch the axis at $(3, 0)$, up through $(4, 4)$ and on; negative for $x < 0$.

d. For $0 < k < 4$ the line $y = k$ meets the curve once in $(0, 1)$, once in $(1, 3)$ and once in $(3, 4)$ (the curve goes from 0 to 4, 4 to 0, 0 to 4 there, unbroken), and a cubic has at most three roots: **three roots exactly when $0 < k < 4$**. $k = 4$: $x = 1$ (a double root, the line touches the hump) and $x = 4$: two. $k = 0$: $x = 0$ and $x = 3$: two. $k = 5$: for $x \leq 4$, $f \leq 4$, so only one root, beyond 4 (about 4.10).

MARKING SCHEME

✓ a. Crosses at 0, touches at 3.
✓ b. The identity expanded and checked.
✓ $f(x) \leq 4$ for $x \leq 4$, equal at 1 and 4.
✓ c. A sketch through $(1, 4)$, touching at $(3, 0)$.
✓ d. $0 < k < 4$; two roots for $k = 4$ and $k = 0$, one for $k = 5$.

4 A hyperbola moved

ANSWER A · 4

Tests: a rational function as $\frac{1}{x}$ moved; multiplying an inequality by something whose sign changes.

a. $\frac{2x+1}{x-1} = \frac{2(x-1)+3}{x-1} = 2 + \frac{3}{x-1}$: $\frac{3}{x}$ moved 1 right and 2 up. Asymptotes $x = 1$, $y = 2$; intercepts $(0, -1)$, $\left(-\frac{1}{2}, 0\right)$; decreasing on $(-\infty, 1)$ and on $(1, \infty)$ separately.

b. $x = 0$ is in her answer, but $y(0) = -1 < 1$. The inequality is $\frac{3}{x-1} > -1$: always true for $x > 1$; for $x < 1$, multiplying by $x - 1 < 0$ turns the sign, $3 < 1 - x$, $x < -2$. **$(-\infty, -2) \cup (1, \infty)$.**

MARKING SCHEME

✓ a. $2 + \frac{3}{x-1}$; asymptotes $x = 1$, $y = 2$.
✓ Intercepts and decreasing on each branch, with a sketch.
✓ b. A number that shows her answer wrong (e.g. 0).
✓ $(-\infty, -2) \cup (1, \infty)$.

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5 How fast do the days lengthen?

ANSWER A · 4

Tests: a sine wave is steepest where it crosses its middle and flat at its tops; reading a model.

a. $L(81) - L(80) = 1.76 \sin \frac{2\pi}{365} \approx 0.0303$ h ≈ 1.8 minutes; $L(172) - L(171) \approx -0.0001$ h, under a second. A sine wave is steepest where it crosses its middle line (the equinox, day 80) and flat at its top (day about 171), so the days lengthen fastest in March and hardly change near the solstice.

b. Where the sine is 0: $t = 80$ and $t = 80 + \frac{365}{2} = 262.5$ (about 21 March and 19 September). Each is a quarter period, about 91 days, from the longest day (day 171): a sine wave goes from its middle to its top in a quarter of its period.

c. $L > 13$ when $\sin\left(\frac{2\pi(t-80)}{365}\right) > \frac{0.86}{1.76} \approx 0.489$. Counting the whole days that satisfy it gives **123 days** (about day 110 to day 232).

MARKING SCHEME

- ✓ a. About 1.8 min and under a second.
- ✓ Steepest at the middle line, flat at the top.
- ✓ b. Days 80 and about 262–263; a quarter period from the longest day.
- ✓ c. About 123 days.

6 A line across the waves

ANSWER HA · 4

Tests: counting intersections from a sketch; symmetry (both sides odd); a bound that limits where to look.

For $|x| > 10$, $\left|\frac{x}{10}\right| > 1 \geq |\sin x|$: no solutions. Both $\sin x$ and $\frac{x}{10}$ are odd, so if x is a solution so is $-x$: the counts on the two sides are equal, and $x = 0$ is one solution.
For $x > 0$: on $(0, \pi)$ the sine starts above the line (it is steeper at 0) and comes down to 0 at π, where the line is 0.31: **one** crossing (about 2.85). On $(\pi, 2\pi)$ the sine is negative and the line positive: none. On $(2\pi, 3\pi)$ the hump reaches 1 at $\frac{5\pi}{2} \approx 7.85$, where the line is 0.785, and is 0 at both ends, where the line is positive: **two** crossings (about 7.07 and 8.42). From 3π to 10 the sine is negative: none.
So 3 positive, 3 negative and 0: **7 solutions**.

MARKING SCHEME

- ✓ None for $|x| > 10$ ($|\sin x| \leq 1$).
- ✓ Odd on both sides: the solutions come in pairs $\pm x$, with 0.
- ✓ One crossing on the first hump, two on the hump around $\frac{5\pi}{2}$.
- ✓ 7 solutions.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	13	15	**28**

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