HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 27

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Ranges that hide a gap

ANSWER HA · 5

Tests: a range comes from the whole domain, holes included; solving $y = g(x)$ for x as a quadratic, where the discriminant decides which y occur.

a. Domain $\mathbb{R} - \{1\}$. There $f(x) = x + 1$, which takes every value except $1 + 1 = 2$ (only $x = 1$ would give it). **Range $\mathbb{R} - \{2\}$.** She simplified without keeping the domain: the graph is the line $y = x + 1$ with a hole at $(1, 2)$.

b. $y(x^2 + 3) = x + 1$ gives $yx^2 - x + (3y - 1) = 0$. For $y = 0$: $x = -1$, real, so 0 is in the range. For $y \neq 0$, a real x needs the discriminant $1 - 4y(3y - 1) \geq 0$, that is $12y^2 - 4y - 1 \leq 0$, $(6y + 1)(2y - 1) \leq 0$: $-\frac{1}{6} \leq y \leq \frac{1}{2}$. **Range $\left[-\frac{1}{6}, \frac{1}{2}\right]$.** At $y = \frac{1}{2}$: $x = 1$ ($g(1) = \frac{2}{4}$); at $y = -\frac{1}{6}$: $x = -3$ ($g(-3) = \frac{-2}{12}$).

The tempting wrong answer is $\mathbb{R}$ ("the denominator is never 0"): that is the domain.

MARKING SCHEME

✓ a. Domain $\mathbb{R} - \{1\}$, range $\mathbb{R} - \{2\}$.
✓ Her mistake: the value 2 belongs to the excluded $x = 1$.
✓ b. The quadratic $yx^2 - x + 3y - 1 = 0$ and the discriminant condition.
✓ $\left[-\frac{1}{6}, \frac{1}{2}\right]$.
✓ Reached at $x = 1$ and $x = -3$.

2 Machines in a row

ANSWER A · 4

Tests: a composite's domain built from the inside out; each machine must accept what it is given.

a. $f(g(h(x))) = \sqrt{\ln(1 - x)}$. Needs $1 - x > 0$ for the log and $\ln(1 - x) \geq 0$ for the root, that is $1 - x \geq 1$: **domain $(-\infty, 0]$**. $\ln(1 - x)$ takes every value in $[0, \infty)$ there, so **range $[0, \infty)$**.

b. $g(f(h(x))) = \ln \sqrt{1 - x}$. Needs $1 - x \geq 0$ for the root and $\sqrt{1 - x} > 0$ for the log: $x < 1$, **domain $(-\infty, 1)$**. $\sqrt{1 - x}$ takes every positive value, so **range $\mathbb{R}$**.

c. He checked only the innermost condition. The square root also needs $\ln(1 - x) \geq 0$: at $x = 0.5$, $\ln 0.5 < 0$ and $\sqrt{\ln 0.5}$ is not real.

MARKING SCHEME

✓ a. $\sqrt{\ln(1 - x)}$, domain $(-\infty, 0]$, range $[0, \infty)$.
✓ b. $\ln \sqrt{1 - x}$, domain $(-\infty, 1)$.
✓ Range $\mathbb{R}$.
✓ c. The root's condition $\ln(1 - x) \geq 0$ missed, shown with a value.

3 Half a parabola and its mirror

ANSWER

Tests: restricting a domain to get an inverse, and the sign that comes from it; where a function meets its inverse.

a. $f(x) = (x - 1)^2 - 1$. On $[1, \infty)$, $x - 1 \geq 0$, so a larger x gives a larger $(x - 1)^2$: one-to-one (increasing). Range $[-1, \infty)$ ($f(1) = -1$, and every larger value is reached).

b. $y = (x - 1)^2 - 1$ gives $x - 1 = \pm\sqrt{y + 1}$. The domain of f has $x \geq 1$, so $x - 1 \geq 0$: the plus sign. **$f^{-1}(x) = 1 + \sqrt{x + 1}$**, domain $[-1, \infty)$. ($1 - \sqrt{x + 1}$ is the inverse of the other half, $x \leq 1$.)

c. By d., the meeting points of an increasing f and f^{-1} are on $y = x$: $x^2 - 2x = x$, $x(x - 3) = 0$. $x = 0$ is not in the domain of f, so only $x = \mathbf{3}$: $f(3) = 3$ and $f^{-1}(3) = 1 + 2 = 3$. ✓

d. (a, b) on both graphs means $f(a) = b$ and $f^{-1}(a) = b$, that is $f(b) = a$. If $a < b$, increasing gives $f(a) < f(b)$, that is $b < a$: impossible. Likewise $a > b$ is impossible. So $a = b$.

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MARKING SCHEME HA · 5

✓ a. One-to-one shown on $[1, \infty)$; range $[-1, \infty)$.
✓ b. $f^{-1}(x) = 1 + \sqrt{x+1}$, domain $[-1, \infty)$.
✓ The plus sign chosen because $x \geq 1$.
✓ c. $x = 3$ only; 0 rejected (not in the domain).
✓ d. $f(a) = b, f(b) = a$ and increasing force $a = b$.

4 A function that undoes itself

ANSWER A · 4

Tests: "its own inverse" as $f(f(x)) = x$, or as $f^{-1} = f$; an equation in a parameter.

a. Solve $y = \dfrac{3x+5}{x+k}$: $x(y-3) = 5 - ky$, so $f^{-1}(x) = \dfrac{-kx+5}{x-3}$. This equals $\dfrac{3x+5}{x+k}$ for every x when $-k = 3$ and $-3 = k$: $k = \mathbf{-3}$. Check: with $f(x) = \dfrac{3x+5}{x-3}$,

$$f(f(x)) = \frac{3 \cdot \dfrac{3x+5}{x-3} + 5}{\dfrac{3x+5}{x-3} - 3} = \frac{9x + 15 + 5x - 15}{3x + 5 - 3x + 9} = \frac{14x}{14} = x. \checkmark$$

b. Its graph is its own mirror image in $y = x$: symmetric about that line.

MARKING SCHEME

✓ a. f^{-1} found, or $f(f(x))$ set equal to x.
✓ $k = -3$.
✓ The check $f(f(x)) = x$.
✓ b. Symmetric about $y = x$.

5 Twice the same machine

ANSWER HA · 4

Tests: working back from a composite to a function; an equation can have more than one answer.

$g(g(x)) = a(ax+b) + b = a^2x + ab + b$. For all x: $a^2 = 4$ and $b(a+1) = 3$.

$a = 2$: $3b = 3$, $b = 1$: $g(x) = 2x + 1$. Check: $2(2x+1) + 1 = 4x + 3$. ✓

$a = -2$: $-b = 3$, $b = -3$: $g(x) = -2x - 3$. Check: $-2(-2x-3) - 3 = 4x + 3$. ✓

$a^2 = 4$ has two roots, and each gives a function, so there are **two** answers; stopping at $a = 2$ (the "square root of 4") misses $-2x - 3$.

MARKING SCHEME

✓ $g(g(x)) = a^2x + ab + b$, coefficients compared.
✓ $g(x) = 2x + 1$.
✓ $g(x) = -2x - 3$.
✓ Both checked; $a = -2$ is why he should go on.

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6 Energies don't add like magnitudes

ANSWER A · 5

Tests: the exponential and its inverse, the logarithm; energies add, magnitudes don't.

a. One step of magnitude multiplies E by $10^{1.5} \approx 31.6$. So one 7.0 releases about 31.6 times a 6.0, more than ten of them, by a factor of about $31.6 \div 10 \approx 3.2$.

b. $10^{1.5 \times 2.8} = 10^{4.2} \approx 15\,800$.

c. $E(7.8) = 10^{16.5} \approx 3.16 \times 10^{16}$ J and $E(7.3) = 10^{15.75} \approx 0.56 \times 10^{16}$ J: together about 3.7×10^{16} J. Then $M = \dfrac{\log_{10} E - 4.8}{1.5} = \dfrac{16.57 - 4.8}{1.5} \approx 7.85$. Magnitudes are logarithms, so adding them would multiply energies; the aftershock carries only about $\dfrac{1}{5.6}$ of the main shock's energy, which raises $\log_{10} E$ by only about 0.07.

MARKING SCHEME

- ✓ a. One 7.0, about 3.2 times as much (31.6 against 10).
- ✓ b. About 15 800 ($10^{4.2}$).
- ✓ c. The energies found and added (about 3.7×10^{16} J).
- ✓ The magnitude about 7.85, by the inverse.
- ✓ Magnitudes are logarithms: they don't add; the smaller shock adds little.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	13	14	**27**

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