Unit 3 · Function

ANSWER KEY · FOR TEACHERS

Full marks 41

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 4

1. natural domain
2. range
3. codomain
4. one-to-one
5. onto
6. inverse
7. composite
8. transcendental

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: f^{-1} is the inverse function: for $f(x) = 2x$, $f^{-1}(x) = \frac{x}{2}$, but $\frac{1}{f(x)} = \frac{1}{2x}$.

iii. ✗ **False**: f is applied first: $(g \circ f)(x) = g(f(x))$.

iv. ✗ **False**: It is not one-to-one on $\mathbb{R}$ ($f(-2) = f(2)$); only on $[0, \infty)$ is its inverse $\sqrt{x}$.

v. ✓ **True**

vi. ✗ **False**: 2^x is never 0: the range is $(0, \infty)$.

Each carries 0.5 mark.

3 Multiple choice

i. **b.** $\mathbb{R} - \{4\}$ — K · 1

Only $x = 4$ makes the denominator 0. $(4, \infty)$ leaves out the numbers below 4 too, which are fine.

ii. **c.** $\log_2 x$ — U · 1

$y = 2^x \Leftrightarrow x = \log_2 y$. $2^{-x} = \frac{1}{2^x}$ is the reciprocal; $\log_x 2$ has the base and the number swapped.

iii. **c.** $x = 3$ — U · 1

$\log_3(x - 2) = 0$ means $x - 2 = 3^0 = 1$, so $x = 3$. $x = 1$ is where $\log_3 x$ crosses; at $x = 2$ the input is 0, outside the domain.

iv. **b.** $\frac{x + 2}{3 - x}$ — A · 1

$y(x + 1) = 3x - 2$, so $x(y - 3) = -y - 2$ and $x = \frac{y + 2}{3 - y}$. The first option is $\frac{1}{f(x)}$.

v. **b.** $[0, 1)$ — A · 1

$f(x) = 1 - \frac{1}{x^2 + 1}$: it is 0 at $x = 0$ and climbs towards 1 without reaching it. So 0 is in and 1 is out.

vi. **c.** $b(c - 1) = d(a - 1)$ — HA · 1

$f(g(x)) = acx + ad + b$ and $g(f(x)) = acx + bc + d$: equal exactly when $ad + b = bc + d$, that is $b(c - 1) = d(a - 1)$. $a = c$, $b = d$ is enough but not needed: $2x + 1$ and $3x + 2$ commute too.

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4 Short answers

a. MARKING SCHEME U · 2

✓ Domain $[-3, 3]$.

✓ Range $[0, 3]$.

b. MARKING SCHEME A · 2

✓ $(x-2)(x-3) > 0$ (strictly).

✓ $(-\infty, 2) \cup (3, \infty)$.

c. MARKING SCHEME U · 3

✓ $g(-4)$ is not real, so $(f \circ g)(-4)$ doesn't exist.

✓ Domain of $f \circ g$: $[0, \infty)$.

✓ Domain of $g \circ f$: $(-\infty, -1] \cup [1, \infty)$.

d. MARKING SCHEME A · 2

✓ Writing the composite in terms of $2x + 3$ (or substituting $x = \frac{u-3}{2}$).

✓ $g(x) = x^2 + 1$.

e. MARKING SCHEME A · 3

✓ $3^y = 2x + 1$.

✓ $g^{-1}(x) = \frac{3^x - 1}{2}$.

✓ Domain $\mathbb{R}$, range $\left(-\frac{1}{2}, \infty\right)$.

5 Structured question

ANSWER A · 5

a. $(f \circ g)(x) = 2(x^2+2) - 3 = 2x^2 + 1$; $(g \circ f)(x) = (2x-3)^2 + 2 = 4x^2 - 12x + 11$. At $x = 0$ they are 1 and 11: different.

b. $2x^2 + 1 = 4x^2 - 12x + 11$ gives $x^2 - 6x + 5 = 0$, $(x-1)(x-5) = 0$: $x = \mathbf{1}$ **or** $x = \mathbf{5}$. Check: $f(g(1)) = f(3) = 3$, $g(f(1)) = g(-1) = 3$. ✓

MARKING SCHEME

✓ a. $(f \circ g)(x) = 2x^2 + 1$.

✓ $(g \circ f)(x) = 4x^2 - 12x + 11$, with a value showing they differ.

✓ b. $x^2 - 6x + 5 = 0$.

✓ $x = 1$ or $x = 5$.

✓ A check.

6 Long question

ANSWER

a. Domain $\mathbb{R} - \{2\}$. $y(x-2) = x + 1$ gives $x = \frac{2y+1}{y-1}$, real for $y \neq 1$: **range $\mathbb{R} - \{1\}$.**

b. $\mathbf{f^{-1}(x) = \frac{2x+1}{x-1}}$ (from a.), $x \neq 1$. $y = 2x - 1$ gives $x = \frac{y+1}{2}$: $\mathbf{g^{-1}(x) = \frac{x+1}{2}}$.

c. Directly: $(g \circ f)(x) = 2 \cdot \frac{x+1}{x-2} - 1 = \frac{x+4}{x-2}$. From $y = \frac{x+4}{x-2}$, $x(y-1) = 2y + 4$, so $(g \circ f)^{-1}(x) = \frac{2x+4}{x-1}$.

By the rule: $f^{-1}\left(\frac{x+1}{2}\right) = \frac{(x+1)+1}{\frac{x+1}{2} - 1} = \frac{x+2}{\frac{x-1}{2}} = \frac{2x+4}{x-1}$. **The same.** ✓

MARKING SCHEME

✓ a. Domain $\mathbb{R} - \{2\}$.

✓ Range $\mathbb{R} - \{1\}$, by solving for x.

✓ b. $f^{-1}(x) = \frac{2x+1}{x-1}$.

✓ $g^{-1}(x) = \frac{x+1}{2}$.

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✓ c. $(g \circ f)(x) = \dfrac{x+4}{x-2}$. **A · 8**

✓ Its inverse $\dfrac{2x+4}{x-1}$.

✓ $f^{-1}(g^{-1}(x))$ set up with g^{-1} first.

✓ Simplified to $\dfrac{2x+4}{x-1}$: the two agree.

7 Think

ANSWER **HA · 3**

a. $\dfrac{E(M+2)}{E(M)} = 10^{1.5(M+2)+4.8-1.5M-4.8} = 10^3 = 1000$.

b. The inverse is $M = \dfrac{\log_{10} E - 4.8}{1.5}$, so doubling E adds $\dfrac{\log_{10} 2}{1.5} \approx \dfrac{0.301}{1.5} \approx 0.2$ to M. Doubling the energy raises the magnitude by only about 0.2.

MARKING SCHEME

✓ a. The ratio $10^{1.5 \times 2} = 1000$.

✓ b. Using the inverse (logarithm): the difference is $\dfrac{\log_{10} 2}{1.5}$.

✓ About 0.2.

8 Activity: A function and its mirror

(not marked)

WHAT STUDENTS SHOULD FIND

2^x: 0.25, 0.5, 1, 2, 4, 8. The points of $\log_2 x$ are the same pairs swapped: (0.25, −2), (0.5, −1), (1, 0), (2, 1), (4, 2), (8, 3), mirror images in $y = x$. 2^x crosses the y-axis at (0, 1) and never meets the x-axis: domain $\mathbb{R}$, range $(0, \infty)$. $\log_2 x$ crosses the x-axis at (1, 0) and never meets the y-axis: domain $(0, \infty)$, range $\mathbb{R}$.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	5	10	22	4	**41**

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