Unit 2 · Real Numbers

HARDER PRACTICE QUESTIONS

Full marks 29

Write your answers on separate paper. Show your working. Write every solution set as an interval or a union of intervals, or list its points.

1 Roots that don't belong [5]

a. Solve $|x^2 - 4| = 3x$. Check every root you keep, and say why each root you reject fails.

b. Bibek solves $|x - 2| = 3x + 6$ by squaring both sides: $(x - 2)^2 = (3x + 6)^2$. He gets two roots. Find them, say which one is not a solution, and explain which equation the other root really solves.

2 Sums and differences of distances [6]

Let $S(x) = |x - 1| + |x - 3|$ and $D(x) = |x - 1| - |x - 3|$.

a. Describe, in words about distances, how many solutions $S(x) = k$ has for each real k. Find them for $k = 5$.

b. Do the same for $D(x) = k$. Find the solutions for $k = 1$ and for $k = 2$.

c. Solve $D(x) \geq 1$.

d. Pasang says: "An equation with two absolute values has at most two solutions." Use a. or b. to show that she is wrong.

3 Absolute values in a fraction [4]

a. Solve $\dfrac{1}{|2x - 3|} > 4$.

b. Solve $\dfrac{|x - 1|}{|x + 2|} \leq 1$, and say what the inequality means about distances.

4 How close is close enough? [5]

a. Show that if $|x - 2| < 1$, then $|x + 2| < 5$.

b. Using $x^2 - 4 = (x - 2)(x + 2)$, show that if $|x - 2| < 0.1$, then $|x^2 - 4| < 0.5$.

c. Find a positive number δ such that $|x - 2| < \delta$ makes $|x^2 - 4| < 0.01$, and prove that yours works.

d. Srijan answers c. with $\delta = 0.0025$, "because $x + 2$ is about 4, and $0.01 \div 4 = 0.0025$". Show that his δ does not quite work.

5 Will the shaft fit? [5]

In a workshop in Butwal, a shaft's diameter d (mm) is made to $|d - 25.00| \leq 0.03$, and the hole of the bearing it fits into to $|h - 25.08| \leq 0.02$. The clearance is $c = h - d$. (Made-up numbers.)

a. Write the possible d and h as intervals.

b. Find the interval of possible clearances, and write it as $|c - m| \leq t$.

c. Explain with the triangle inequality why the tolerance of c is the sum of the two tolerances.

d. The bearing needs a clearance of at least 0.04 mm. Will every shaft and hole fit? If not, how small must the shaft's tolerance be, the hole's staying as it is?

6 Between Baudhāyana and \(\sqrt{2} \) [4]

Baudhāyana's value of $\sqrt{2}$ was $\frac{577}{408} = 1.4142156\ldots$, and $\sqrt{2} = 1.4142135\ldots$.

a. Write down a rational number and an irrational number that both lie strictly between $\sqrt{2}$ and $\frac{577}{408}$. Say why each is rational or irrational.

b. Let $a < b$ be any two rational numbers. Show that $a + \frac{b-a}{\sqrt{2}}$ lies strictly between a and b, and that it is irrational. (Use that $\sqrt{2}$ is irrational.)