Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Roots that don't belong

ANSWER A · 5

Tests: a case's root must lie in its own piece, and the right side must not be negative; squaring brings in roots of $|f| = -g$.

a. The left side is never negative, so $3x \geq 0$: $x \geq 0$.

Where $x^2 \geq 4$ (with $x \geq 0$: $x \geq 2$): $x^2 - 4 = 3x$, $x^2 - 3x - 4 = 0$, $(x-4)(x+1) = 0$: $x = 4$ is kept; $x = -1$ is rejected ($3x = -3 < 0$).

Where $x^2 < 4$ ($0 \leq x < 2$): $4 - x^2 = 3x$, $x^2 + 3x - 4 = 0$, $(x+4)(x-1) = 0$: $x = 1$ is kept; $x = -4$ is rejected (not in the piece, and $3x < 0$).

$x = 1$ or $x = 4$. Check: $|1 - 4| = 3 = 3(1)$; $|16 - 4| = 12 = 3(4)$.

The tempting wrong answer keeps all four roots, $-4, -1, 1, 4$: for $x = -1$, $|1 - 4| = 3$ but $3x = -3$.

b. $(x-2)^2 - (3x+6)^2 = 0$ factors as $(x - 2 - 3x - 6)(x - 2 + 3x + 6) = (-2x - 8)(4x + 4) = 0$: $x = -4$ or $x = -1$. $x = -1$ works ($|-3| = 3 = 3(-1) + 6$). $x = -4$ does not: $|-6| = 6$ but $3(-4) + 6 = -6$. Squaring loses the sign: $x = -4$ solves $|x - 2| = -(3x + 6)$. After squaring, every root must be checked in the first equation.

MARKING SCHEME

- ✓ a. The condition $x \geq 0$ (or a check of the sign of $3x$).
- ✓ Both cases set up, with their pieces.
- ✓ $x = 1$ and $x = 4$ kept; -1 and -4 rejected, with the reason for each.
- ✓ b. The roots -4 and -1.
- ✓ -4 rejected: it solves $|x - 2| = -(3x + 6)$; squaring can't tell a number from its negative.

2 Sums and differences of distances

ANSWER HA · 6

Tests: $|x - a|$ as a distance; a sum of distances is constant between the points, a difference constant outside them; the number of solutions changes with k.

a. $S(x)$ is the distance from x to 1 plus the distance to 3. Between 1 and 3 it is always 2; outside it grows by 2 for each unit step away. So $S(x) = k$ has **no** solution for $k < 2$, **every** $x \in [1, 3]$ for $k = 2$, and **two** solutions, $x = 2 \pm \frac{k}{2}$, for $k > 2$. For $k = 5$: $x = -\frac{1}{2}$ or $x = \frac{9}{2}$.

b. $D(x)$ is how much farther x is from 1 than from 3. Left of 1 it is $(1 - x) - (3 - x) = -2$; right of 3 it is $(x - 1) - (x - 3) = 2$; between, $D(x) = 2x - 4$, rising from -2 to 2. So $D(x) = k$ has no solution for $|k| > 2$, one solution $x = 2 + \frac{k}{2}$ for $-2 < k < 2$, and a whole ray for $k = \pm 2$: $(-\infty, 1]$ for $k = -2$, $[3, \infty)$ for $k = 2$. For $k = 1$: $2x - 4 = 1$, $x = \frac{5}{2}$. For $k = 2$: every $x \geq 3$, $[3, \infty)$.

c. Between 1 and 3, $2x - 4 \geq 1$ gives $x \geq \frac{5}{2}$; right of 3, $D = 2 \geq 1$ always; left of 1, $D = -2$. $\left[\frac{5}{2}, \infty\right)$.

d. $|x - 1| + |x - 3| = 2$ has infinitely many solutions (all of $[1, 3]$); so does $|x - 1| - |x - 3| = 2$.

MARKING SCHEME

- ✓ a. The sum is 2 between 1 and 3, larger outside: none, an interval, two.
- ✓ $k = 5$: $-\frac{1}{2}$, $\frac{9}{2}$.
- ✓ b. $D = -2$, $2x - 4$, 2 on the three pieces, and the count for each k.
- ✓ $k = 1$: $\frac{5}{2}$; $k = 2$: $[3, \infty)$.
- ✓ c. $\left[\frac{5}{2}, \infty\right)$.
- ✓ d. A counterexample with infinitely many solutions.

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3 Absolute values in a fraction

ANSWER A · 4

Tests: multiplying by a positive absolute value keeps the sign, but the point where it is 0 must be thrown out; reading a quotient of distances.

a. The fraction is defined only for $x \neq \frac{3}{2}$, and then $|2x-3| > 0$, so multiplying keeps the sign: $0 < |2x-3| < \frac{1}{4}$. Then $-\frac{1}{4} < 2x-3 < \frac{1}{4}$, $\frac{11}{8} < x < \frac{13}{8}$, without $x = \frac{3}{2}$: $\left(\mathbf{\frac{11}{8}}, \mathbf{\frac{3}{2}}\right) \cup \left(\mathbf{\frac{3}{2}}, \mathbf{\frac{13}{8}}\right)$. The tempting wrong answer is $\left(\frac{11}{8}, \frac{13}{8}\right)$, which keeps the point where the fraction is not defined.

b. For $x \neq -2$, it says $|x-1| \leq |x+2|$: x is at least as close to 1 as to -2. The point halfway is $-\frac{1}{2}$, so $\left[-\mathbf{\frac{1}{2}}, \infty\right)$ (-2 is not in it anyway). Check by squaring: $x^2 - 2x + 1 \leq x^2 + 4x + 4$ gives $6x \geq -3$, $x \geq -\frac{1}{2}$.

MARKING SCHEME

✓ a. $0 < |2x-3| < \frac{1}{4}$.

✓ The union of two intervals, $\frac{3}{2}$ left out.

✓ b. $|x-1| \leq |x+2|$, read as nearer to 1 than to -2.

✓ $\left[-\frac{1}{2}, \infty\right)$.

4 How close is close enough?

ANSWER HA · 5

Tests: a neighborhood written with absolute values; $|xy| = |x||y|$ and the triangle inequality used to bound; the tempting estimate "$x + 2 \approx 4$".

a. $|x+2| = |(x-2)+4| \leq |x-2| + 4 < 1 + 4 = 5$ (the triangle inequality). (Or: $1 < x < 3$, so $3 < x + 2 < 5$.)

b. $|x-2| < 0.1$ gives $3.9 < x + 2 < 4.1$, so $|x^2 - 4| = |x-2||x+2| < 0.1 \times 4.1 = 0.41 < 0.5$.

c. Take $\delta = 0.002$ (any δ with $\delta \leq 1$ and $5\delta \leq 0.01$ does). If $|x-2| < 0.002$, then $|x-2| < 1$, so $|x+2| < 5$ by a., and $|x^2 - 4| < 0.002 \times 5 = 0.01$. ✓ (Exactly: $|x-2| < \delta$ gives $|x^2 - 4| < \delta(4+\delta)$, so any $\delta \leq \sqrt{4.01} - 2 \approx 0.0024984$ works.)

d. $x + 2$ is a little **more** than 4 when $x > 2$. Take $x = 2.00249999$: $|x-2| < 0.0025$, but $x^2 - 4 \approx 0.0100062 > 0.01$.

MARKING SCHEME

✓ a. $|x+2| < 5$ shown (triangle inequality or the interval).

✓ b. $|x^2 - 4| = |x-2||x+2| < 0.1 \times 4.1 < 0.5$.

✓ c. A $\delta \leq 0.0024984$ (e.g. 0.002).

✓ The proof that it works (bounding $|x+2|$).

✓ d. A number within 0.0025 of 2 with $x^2 - 4 > 0.01$.

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5 Will the shaft fit?

ANSWER A · 5

Tests: center and radius of an interval; the triangle inequality as "tolerances add"; using an interval to decide something.

a. $d \in [24.97, 25.03]$, $h \in [25.06, 25.10]$.

b. The smallest clearance is the smallest hole with the largest shaft: $25.06 - 25.03 = 0.03$; the largest is $25.10 - 24.97 = 0.13$. $c \in [0.03, 0.13]$: $|c - 0.08| \leq 0.05$.

c. $c - 0.08 = (h - 25.08) - (d - 25.00)$, so $|c - 0.08| \leq |h - 25.08| + |d - 25.00| \leq 0.02 + 0.03 = 0.05$.

d. No: a 25.03 mm shaft in a 25.06 mm hole leaves 0.03 mm. With shaft tolerance t, the smallest clearance is $25.06 - (25.00 + t) = 0.06 - t$; $0.06 - t \geq 0.04$ needs $t \leq 0.02$ mm.

MARKING SCHEME

- ✓ a. Both intervals.
- ✓ b. [0.03, 0.13], from the extreme pairs.
- ✓ $|c - 0.08| \leq 0.05$.
- ✓ c. The triangle inequality on $(h - 25.08) - (d - 25.00)$.
- ✓ d. Not every pair; $t \leq 0.02$ mm.

6 Between Baudhāyana and \(\sqrt{2} \)

ANSWER HA · 4

Tests: density of the rationals and the irrationals; a proof by contradiction with the closure of the rationals.

a. Rational: $1.414214 = \dfrac{1414214}{1000000}$ (it ends). Irrational: $\sqrt{2} + 0.000001 = 1.4142145\ldots$ (if it were rational, $\sqrt{2}$ would be a rational minus a rational, rational). Both lie between $1.4142135\ldots$ and $1.4142156\ldots$. (Any decimal that ends, from 1.414214 to 1.414215, works.)

b. $0 < \dfrac{1}{\sqrt{2}} < 1$ and $b - a > 0$, so $0 < \dfrac{b-a}{\sqrt{2}} < b - a$: adding a, the number lies strictly between a and b. If it were a rational r, then $\sqrt{2} = \dfrac{b-a}{r-a}$ (with $r \neq a$) would be rational, which it isn't. So it is irrational.

MARKING SCHEME

- ✓ a. A rational between them, with the reason.
- ✓ An irrational between them, with the reason.
- ✓ b. Strictly between, from $0 < \dfrac{1}{\sqrt{2}} < 1$.
- ✓ Irrational: otherwise $\sqrt{2} = \dfrac{b-a}{r-a}$ would be rational.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	14	15	**29**

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