Unit 2 · Real Numbers

ANSWER KEY · FOR TEACHERS

Full marks 42

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 4

1. irrational
2. real line
3. closed
4. open
5. unbounded
6. absolute value
7. neighborhood
8. triangle inequality

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: ∞ is not a number, so its bracket is always round: $[2, \infty)$.

iii. ✗ **False**: $\sqrt{x^2} = |x|$: for $x = -3$, $\sqrt{9} = 3$, not -3.

iv. ✗ **False**: It holds for no x: a distance is never negative. ($|x| > -2$ holds for every x.)

v. ✓ **True**

vi. ✗ **False**: Only when a, b have the same sign (or one is 0): $|3 + (-5)| = 2$, but $|3| + |-5| = 8$.

Each carries 0.5 mark.

3 Multiple choice

i. **c.** 5 but not -2 — K · 1

A round bracket leaves its end out, a square one takes it in.

ii. **c.** $|x|$ — U · 1

A square root is never negative, so $\sqrt{x^2} = |x|$: for $x = -4$, $\sqrt{16} = 4$, not -4. $\pm x$ is two numbers, and a square root is one.

iii. **b.** $|x - 2| \leq 3$ — U · 1

The center is the midpoint $\frac{-1 + 5}{2} = 2$ and the radius half the length, $\frac{6}{2} = 3$; both ends are in, so $\leq$.

$|x - 2| < 3$ is the open interval.

iv. **c.** 5 — A · 1

$-1 < x < 5$: the integers 0, 1, 2, 3, 4. Counting -1 and 5 as well gives 7, but the ends are out.

v. **b.** $x = 1$ — A · 1

x is as far from -3 as from 5: the midpoint, $\frac{-3 + 5}{2} = 1$. (Squaring both sides gives $6x + 9 = -10x + 25$, $x = 1$ too.) 4 is half the distance from -3 to 5, not the midpoint.

vi. **c.** infinitely many — HA · 1

The distances from x to 0 and to 4 add to 4 exactly when x lies between them: every $x \in [0, 4]$. Outside, the sum is more than 4. (Example 4 had the same shape: the sum is constant between the two points.)

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4 Short answers

a. MARKING SCHEME U · 2

- ✓ a. $|x - 2| \leq 5$.
- ✓ b. $|x - 3| > 2$.

b. MARKING SCHEME A · 2

- ✓ $3x + 1 = 8$ or $3x + 1 = -8$.
- ✓ $x = \frac{7}{3}$ or $x = -3$.

c. MARKING SCHEME U · 3

- ✓ The mistake: multiplying by -1 without turning the signs round.
- ✓ $1 < x < 5$, the interval $(1, 5)$.
- ✓ A check with a number (e.g. 3 works, -3 doesn't).

d. MARKING SCHEME A · 3

- ✓ The two cases ($x \geq 2, x < 2$) or the condition $3x + 6 \geq 0$.
- ✓ $x = -4$ rejected, with the reason.
- ✓ $x = -1$, checked.

e. MARKING SCHEME A · 3

- ✓ Writing $x = (x - y) + y$.
- ✓ The triangle inequality: $|x| \leq |x - y| + |y|$.
- ✓ Rearranged to $|x - y| \geq |x| - |y|$.

5 Structured question

ANSWER U · 5

a. A: a filled dot at -3, an open dot at 2, shaded between. B: an open dot at 0, a filled dot at 5, shaded between.

b. $A \cup B = [-3, 5]$; $A \cap B = (0, 2)$; $B - A = [2, 5]$ (2 is in B and not in A).

c. $A \cap B \cap C = (0, 2) \cap (-\infty, 1] = (0, 1]$; $\overline{C} = (1, \infty)$.

MARKING SCHEME

- ✓ a. Both intervals with the right dots (filled at -3 and 5, open at 2 and 0).
- ✓ b. $A \cup B = [-3, 5]$ and $A \cap B = (0, 2)$.
- ✓ $B - A = [2, 5]$.
- ✓ c. $A \cap B \cap C = (0, 1]$.
- ✓ $\overline{C} = (1, \infty)$.

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6 Long question

ANSWER A · 8

a. For $x < 1$: $(1 - x) + (3 - x) = 4 - 2x \leq 4$ gives $x \geq 0$, so $0 \leq x < 1$.
For $1 \leq x \leq 3$: $(x - 1) + (3 - x) = 2 \leq 4$, true for all these x.
For $x > 3$: $(x - 1) + (x - 3) = 2x - 4 \leq 4$ gives $x \leq 4$, so $3 < x \leq 4$.
Together: **[0, 4]**.

b. $|x - 1| + |x - 3|$ is the distance from x to 1 plus the distance to 3. Between 1 and 3 the sum is always 2, the gap between them. Outside, each unit step away adds 2 to the sum, so it reaches 4 one unit beyond each end: at 0 and at 4.

c. The sum is never less than 2, and equals 2 exactly between 1 and 3: $|x - 1| + |x - 3| \leq 2$ has solution set $[1, 3]$, and $|x - 1| + |x - 3| \leq 1$ has **no** solution ($\emptyset$).

MARKING SCHEME

- ✓ a. The piece $x < 1$: $4 - 2x \leq 4$, so $0 \leq x < 1$.
- ✓ The piece $1 \leq x \leq 3$: the sum is 2, so all of it.
- ✓ The piece $x > 3$: $2x - 4 \leq 4$, so $3 < x \leq 4$.
- ✓ The union $[0, 4]$.
- ✓ b. The sum as the distances to 1 and to 3.
- ✓ Constant 2 between them, growing by 2 a unit outside, so 4 at 0 and 4.
- ✓ c. $[1, 3]$.
- ✓ No solution, as the sum is never below 2.

7 Think

ANSWER HA · 3

She is right: $|(x + y) - 8| = |(x - 3) + (y - 5)| \leq |x - 3| + |y - 5| < 0.1 + 0.1 = 0.2$, by the triangle inequality. It can be 0.19: $x = 3.095$, $y = 5.095$ are each within 0.1, and $x + y - 8 = 0.19$.

MARKING SCHEME

- ✓ $(x + y) - 8$ written as $(x - 3) + (y - 5)$.
- ✓ The triangle inequality gives less than 0.2: she is right.
- ✓ An example reaching 0.19 (e.g. $x = 3.095$, $y = 5.095$).

8 Activity: Center and radius

(not marked)

WHAT STUDENTS SHOULD FIND

$|x - 4| \leq 3$: $[1, 7]$, center 4, radius 3, in. $|x + 1| < 2$: $(-3, 1)$, center -1, radius 2, out. $|2x - 6| \leq 4$ is $|x - 3| \leq 2$: $[1, 5]$, center 3, radius 2, in. $(-2, 8)$: $|x - 3| < 5$, center 3, radius 5, out. The center is the midpoint of the ends, the radius half the interval's length; square brackets go with $\leq$, round ones with $<$.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	5	15	18	4	**42**

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