Unit 21 · Numerical Integration

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **Option**: Area 7 is Computational Methods **or** Mechanics, each school one; this unit and Unit 20 are Computational Methods
- **CDC's grid (2078):** Computational Methods (Units 20 and 21 together): 6 of 75 marks, one 1-mark choice and one 5-mark short question
- **CDC's model paper:** the short question split: 2 marks on bisection, 3 marks for the trapezoidal rule with $n = 4$
- **MCQ (1 mark)**
  - the trapezoidal formula; what it puts on each strip; how many ordinates for $h = 0.5$
  - too big or too small (e^x); when Simpson's can't be used; a relative error
  - the error with strips halved; $n = 2$ answers; which integral Simpson's gets exactly; $(4T_4 - T_2)/3$
- **Short (5 marks, in parts)**
  - $\int_1^2 dx/x$, $n = 4$, both rules, compared with the exact value
  - the day's sunlight on a roof from hourly values, by Simpson's
  - a plot's area from offsets to a stream, in m^2 and ropani

Key terms

Numerical integration: the area from strips that fit the curve, where there is no anti-derivative to use.

Strip width $h = (b - a)/n$, n strips.

Ordinates $y_0, y_1, \ldots, y_n$: the heights at the cuts, $y_k = f(a + kh)$; n strips have $n + 1$.

Error = exact value − approximate value, where the exact one can be found.

- needed when the curve is only **measured** (a river's depths, a field's width) or has no anti-derivative we know (e^{-x^2})

Formulas

$\frac{h}{2}[(y_0 + y_n) + 2(y_1 + \ldots + y_{n-1})]$	any n **Use:** trapezoidal rule
$\frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$	n **even** **Use:** Simpson's rule
$\frac{h}{3}(y_0 + 4y_1 + y_2)$	one pair of strips **Use:** area under one parabola

Trapezoidal rule

1. chords join neighboring tops: each strip a trapezium
2. first strip: $(h/2)(y_0 + y_1)$; and so on
3. add: ends once, every inside ordinate twice (shared by two strips)

- half the width × (first and last + twice all the rest)
- the average of the left-height and right-height rectangle sums
- $n = 4$: **five** ordinates; "five ordinates" means $n = 4$

- bends up (concave up, e^x): chords above → **too big**; bends down: too small
- π: $\int_0^1 dx/(1+x^2)$, $n=4$: 0.7828; exact $\pi/4 =$ 0.785398; error 0.0026

Simpson's rule

One parabola through $(-h, y_0)$, $(0, y_1)$, (h, y_2)

1. $y_0 = Ah^2 - Bh + C$, $y_1 = C$, $y_2 = Ah^2 + Bh + C$
2. $\int_{-h}^{h} (Ax^2 + Bx + C)\,dx = (h/3)(2Ah^2 + 6C)$ (Bx odd: cancels)
3. $y_0 + y_2 = 2Ah^2 + 2C \to 2Ah^2 + 6C = y_0 + 4y_1 + y_2$

- pairs of strips: odd ordinates ×4 (middles), even inside ×2 (shared ends)
- weights **1, 4, 2, 4, 2, …, 4, 1**, × $h/3$: Simpson's **one-third rule**
- n even: an odd number of ordinates (5, 7, 9…); 6 ordinates → trapezoidal, or Simpson's on four strips + a trapezium
- **exact for every cubic** (so $n=2$ gives $\int x^3$ exactly)
- π: the same five numbers → 0.785392, error 0.000006: over 400 times better

How good

Strips n	2	4	8	16
Trapezoidal error	0.4292	0.1039	0.0258	0.0064
Simpson's error	−0.0944	−0.0046	−0.00027	−0.000017

- for $\int_0^{\pi} \sin x\,dx = 2$
- halving h: trapezoidal error ÷ about **4** (h^2); Simpson's ÷ about **16** (h^4)
- $(4T_4 - T_2)/3$ = Simpson's rule with 4 strips

	Trapezoidal rule	Simpson's rule
curve replaced by	a chord on each strip	a parabola on each pair of strips
strips n	any	even
weights	1, 2, 2, …, 2, 1, × $h/2$	1, 4, 2, 4, …, 4, 1, × $h/3$
halving h divides the error by	about 4	about 16

The river: depths across, eleven soundings, $h = 2$ m

- Simpson's: cross-section **31.6 m²**; the two rules agree within 0.2 m²
- flow ≈ area × speed = 31.6 × 1.5 ≈ **47 m³/s**

COMMON MISTAKES

✗ $n = 4$ taken as four ordinates
✓ four strips, five ordinates.

✗ forgetting the 2 on the inside ordinates
✓ ends once, the rest twice.

✗ Simpson's with an odd n
✓ n must be even.

✗ Simpson's weights 1, 2, 4, …
✓ 1, 4, 2, 4, …, 4, 1: odd ordinates ×4.

✗ $h/2$ in Simpson's
✓ $h/3$.

✗ $h = (b-a)/(n+1)$
✓ $(b-a)/n$.

✗ error as approximate − exact
✓ exact − approximate.