Unit 20 · Numerical Computation

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **Option**: Area 7 is Computational Methods **or** Mechanics, each school one; this unit and Unit 21 are Computational Methods, Units 22–23 Mechanics
- **CDC's grid (2078):** Computational Methods (Units 20 and 21 together): 6 of 75 marks, one 1-mark choice and one 5-mark short question; no long question
- **MCQ (1 mark)**
  - when bisection can start; the Newton–Raphson formula; which equation is transcendental
  - the half after one bisection step; when Newton–Raphson fails; an approximate error
  - the error bound after 5 halvings; steps for error < 0.01; one Newton step for $x^2 - 5 = 0$
  - Newton for $1/x - a$; bisection on $\tan x$ over $[1, 2]$
- **Short (5 marks)**
  - a root between 2 and 3, to two decimal places, by bisection
  - $x \log_{10} x = 1.2$ to four places by Newton–Raphson from 2.5
  - Newton's cycle for $x^3 - 2x + 2$

Key terms

Root of $f(x) = 0$: a number that makes it true; where $y = f(x)$ meets the X-axis.

Algebraic equation: $f(x)$ a polynomial (or can be made one: $\sqrt{x} = x - 2$).

Transcendental: a trigonometric, exponential or logarithmic function of x ($x = \cos x$, $e^{-x} = x$).

Error of x_n: $\alpha - x_n$; **absolute error** $|\alpha - x_n|$; **relative** $|(\alpha - x_n)/\alpha|$ ($\times 100$: **percentage**).

Approximate error $E_a = |x_{n+1} - x_n|$; **relative approximate error** $\varepsilon_a = |(x_{n+1} - x_n)/x_{n+1}| \times 100\%$.

Tolerance: the error at which to stop.

- no formula of roots for degree 5 or more (Abel, 1824); none for transcendental equations → **numerically**
- **locate**: f continuous on $[a, b]$, $f(a), f(b)$ **opposite signs** → a root between (intermediate value theorem, Unit 17)
- **radians** always: in degree mode $\cos 1 = 0.99985$
- stop when E_a or ε_a is below the tolerance, or two values agree to k places

Formulas

$c = (a + b)/2$	keep the half with the sign change **Use:** bisection
$\lvert\alpha - c_n\rvert \le (b - a)/2^n$	known before you start **Use:** bisection error
$2^n > (b - a)/E$	E the tolerance **Use:** steps needed
$x_{n+1} = x_n - f(x_n)/f'(x_n)$	$f'(x_n) \neq 0$ **Use:** Newton–Raphson

Bisection

1. midpoint $c = (a + b)/2$, find $f(c)$
2. $f(c) = 0$: the root; $f(a), f(c)$ opposite: $b = c$; else $a = c$
3. repeat; for k places, stop when a and b round the same

- table of steps: n, a, b, c, sign of $f(c)$
- width 1, error < 0.001: $2^{10} = 1024 \rightarrow 10$ steps
- its approximate error $(b - a)/2^{n+1}$: half the bound
- slow (about 3.3 steps a decimal place) but **certain** for a continuous f; needs only signs, no derivative
- $x = \cos x$ on $(0, 1)$: 20 halvings → 0.7390852

Newton–Raphson

1. tangent at $(x_0, f(x_0))$, slope $f'(x_0)$
2. $y - f(x_0) = f'(x_0)(x - x_0)$
3. $y = 0$: $x_1 = x_0 - f(x_0)/f'(x_0)$
4. repeat

- near a root the correct places about **double** each step
- $x = \cos x$ from 1: $f = x - \cos x$, $f' = 1 + \sin x \to 0.7390851332$ in five steps
- $x^2 - 5$ from 2: $2 - (-1)/4 = 2.25$

When it fails

- **flat tangent**: $f'(x_n) = 0$, divides by 0; near-flat throws the guess far
- **a cycle**: $x^3 - 2x + 2$ from 0: 0, 1, 0, 1, … (root near -1.77)
- **running away**: $\tan^{-1} x$ from 1.5: -1.69, 2.32, -5.11, 32.3; from 1 it closes in
- remedy: locate with a sign change first; a few bisection steps if the guesses jump
- spreadsheet: A2 `=A1-(A1-COS(A1))/(1+SIN(A1))`, fill down

The loan: Rs 1,00,000, 24 payments of Rs 4,800

- $4800(1 - (1 + r)^{-24})/r = 100000$; Newton–Raphson $\to r = 0.01164$, 1.164% a month
- $12 \times 1.164\% \approx$ **13.97%** a year (not the 7.6% of 15,200 ÷ 1,00,000); with interest on interest 14.9%
- **false root**: multiplying by r made $r = 0$ a root; $f' = 0$ near 0.0057; a start at 0.005 runs to 0
- check the answer makes sense; start on the right side of a turning point

	Bisection	Newton–Raphson
needs	signs of f, a sign change	f', a good start
speed	slow, error halves each step	fast near a root, places double
sure?	yes, for a continuous f	may fail: flat tangent, cycle, runaway

COMMON MISTAKES

✗ calculator in degrees
✓ RAD.

✗ bisection across a break ($\tan x$ on [1, 2])
✓ closes in on $\pi/2$, not a root: f must be continuous.

✗ keeping the half without the sign change
✓ keep the half whose ends differ in sign.

✗ $x_{n+1} = x_n + f/f'$
✓ minus.

✗ stopping when $f(x_n)$ looks small
✓ stop on the approximate error or agreeing places.

✗ error bound $(b - a)/n$
✓ $(b - a)/2^n$.

✗ every root Newton gives is the wanted one
✓ check it makes sense (the loan's $r = 0$).