Unit 19 · Anti-derivatives

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Calculus (Units 17–19 together): 22 of 75 marks, four 1-mark choices, two 5-mark short questions and one 8-mark long question; integration appears in all three
- **MCQ (1 mark)**
  - a basic integral ($\sec^2 x$, $\tan x$, $\ln x$, xe^{x^2}); $\int f(3x)\,dx$
  - $\int_a^b f\,dx$ from F; an odd function over $[-a, a]$; ranges added
  - the area under $y = x^2$; between $y = x^2$ and $y = 4$; $\int e^x(\sin x + \cos x)\,dx$
- **Short (5 marks, in parts)**
  - integrate; by substitution; by parts; $\int e^x \cos x\,dx$
  - definite integrals; the water filled from a falling rate of flow
  - the area between $y = x^2 - 2x$ and the X-axis; between $y^2 = 4x$ and $x^2 = 4y$
- **Long (8 marks)**
  - the region between $y = 4x - x^2$ and $y = x$: sketch, meeting points, area

Key terms

Anti-derivative (primitive) F of f: $F'(x) = f(x)$; finding it is **integration**.

Indefinite integral: $\int f(x)\,dx = F(x) + C$, the whole family.

Integrand $f(x)$; **constant of integration** C; dx names the variable.

Definite integral: $\int_a^b f(x)\,dx = F(b) - F(a)$, a **number**; lower limit a, upper limit b.

Signed area: above the axis minus below it.

- two anti-derivatives differ by a constant: $(G - F)' = 0 \to$ constant
- the family: one curve slid up and down, tangents parallel at each x; a point fixes C
- **check by differentiating**: the answer's derivative must be the integrand

Basic integrals

$\int \ldots\,dx$	$= \ldots + C$	$\int \ldots\,dx$	$= \ldots + C$
$x^n\ (n \neq -1)$	$x^{n+1}/(n+1)$	$1/x$	$\ln\lvert x\rvert$
e^x	e^x	a^x	$a^x/\ln a$
$\sin x$	$-\cos x$	$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$	$\csc^2 x$	$-\cot x$
$\sec x \tan x$	$\sec x$	$\csc x \cot x$	$-\csc x$
$1/\sqrt{1-x^2}$	$\sin^{-1} x$	$1/(1+x^2)$	$\tan^{-1} x$
$1/(\lvert x\rvert\sqrt{x^2-1})$	$\sec^{-1} x$	$\tan x$	$\ln\lvert\sec x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$	$\sec x$	$\ln\lvert\sec x + \tan x\rvert$
$\sin^2 x$	$x/2 - \sin 2x/4$	$\cos^2 x$	$x/2 + \sin 2x/4$
$\ln x$	$x \ln x - x$	$f'(x)/f(x)$	$\ln\lvert f(x)\rvert$

- **$\ln|x|$**: for $x < 0$, $(\ln(-x))' = 1/x$; fills the gap at $n = -1$
- $\int kf = k\int f$; $\int(f \pm g) = \int f \pm \int g$; one C at the end
- **no** product or quotient rule: multiply out, simplify, or substitute / by parts
- rewrite first: roots and fractions as powers; an identity ($\tan^2 x = \sec^2 x - 1$); divide top by bottom
- $\int 1/x^2\,dx = -1/x$, not $\ln x^2$
- velocity $\to$ position: integrate, then a starting value fixes C

Substitution: the chain rule backwards

$\int f(g(x))g'(x)\,dx = \int f(u)\,du, \qquad u = g(x),\ du = g'(x)\,dx$

- why: $(F(g(x)))' = f(g(x))g'(x)$
- replace every x; integrate in u; put $g(x)$ back
- the inside's derivative standing outside, maybe short of a constant
- **linear inside**: $\int f(ax+b)\,dx = (1/a)F(ax+b) + C$
  - $\int \cos 3x\,dx = (1/3)\sin 3x$; $\int e^{-2x}dx = -(1/2)e^{-2x}$; $\int dx/(2x+5) = (1/2)\ln\lvert 2x+5\rvert$
- **top = bottom's derivative**: $\int f'/f\,dx = \ln|f| + C$; $\int 2x/(x^2+1)\,dx = \ln(x^2+1)$

- **power × derivative**: $\int [f(x)]^n f'(x)\,dx = [f(x)]^{n+1}/(n+1) + C,\ n \neq -1$
- trig squares and products: double-angle and product formulas first ($\sin^2 x = (1-\cos 2x)/2$)
- standard forms ($1/(x^2+a^2)$, …) and partial fractions: Grade 12

By parts: the product rule backwards

$$\int u\,v\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

1. $(uw)' = uw' + wu'$
2. integrate: $uw = \int uw'\,dx + \int wu'\,dx$
3. rearrange; $v = w'$

- short form: $\int u\,dv = uv - \int v\,du$
- u gets simpler when differentiated; v must be integrable
- **LIATE**: Logarithmic, Inverse trig, Algebraic, Trig, Exponential (ILATE differs only for log vs inverse trig)
  - $x\sin x$: $u = x$; $x\ln x$: $u = \ln x$
- alone ($\ln x$, $\tan^{-1} x$, $\sin^{-1} x$): take $v = 1$
- $x^2\,e^x$: by parts twice
- the integral comes back ($e^x\sin x$, $e^x\cos x$): twice, then solve for it
- $\int e^x(f + f')\,dx = e^x f + C$

The definite integral

- total change from a rate: C cancels; motorbike $s(6) - s(2) = 44$ m
- substituting: **change the limits** and stay in u

Properties

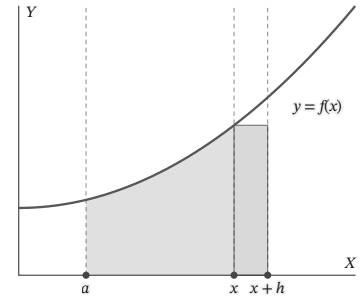
1. $\int_a^a f\,dx = 0$
2. $\int_b^a f\,dx = -\int_a^b f\,dx$
3. $\int_a^b = \int_a^c + \int_c^b$
4. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$
5. $\int_a^b (kf \pm g) = k\int_a^b f \pm \int_a^b g$
6. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\ (t = a - x)$
7. over $[-a, a]$: **odd** $\to 0$; **even** $\to 2\int_0^a f\,dx$

- $\int_{-2}^{2} (x^3 - 4x + \sin x)\,dx = 0$: every term odd

The integral as an area

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Sigma_{k=1}^{n} f(x_k)\,\Delta x$$

- strips of width $\Delta x = (b-a)/n$
- $f \geq 0$: the area bounded by the curve, the X-axis and $x = a$, $x = b$
- $y = x^2$ on $[0, 2]$: left sums 1.75, 2.28, 2.6268; right 3.75, 3.08, 2.7068 $\to 8/3$
- $\int$ is a long S (sum); dx what became of Δx



Fundamental theorem

1. strip: $A(x+h) - A(x)$
2. $mh \leq$ strip $\leq Mh$
3. $h \to 0$: $A'(x) = f(x)$
4. $A(a) = 0 \to A(b) = F(b) - F(a)$

- the rate at which the area grows is the curve's height: slopes and areas joined
- **below the axis**: the integral counts negative; for an area, split at the crossings, add the sizes
  - $y = x^2 - 2x$ on $[0, 2]$: integral $-4/3$, area $4/3$

Area between two curves

$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx \qquad (f \text{ on top})$$

1. sketch; solve $f(x) = g(x)$ for the limits
2. which is on top in each piece (test one value)
3. top minus bottom over each piece; add

- works below the axis too, so long as f stays on top
- curves that change places: split where they cross

The sunlight on the roof: energy = area under the power–time graph

- model $P(t) = 856\sin(\pi t/12.75)$, $0 \leq t \leq 12.75$ h from 5:45
- $E = 856 \cdot (12.75/\pi) \cdot 2 \approx 6948$ Wh ≈ 6.9 kWh per m^2
- PVGIS's hourly bars (rectangles 1 h wide): **6.3 kWh**; the model is 10% high (low sun at the ends)
- any total of a changing rate is an area: energy from power, distance from speed, water from flow

COMMON MISTAKES

✗ no + C on an indefinite integral
✓ always + C.

✗ $\int 1/x\,dx = x^0/0$
✓ $\ln|x| + C$.

✗ $\int fg\,dx = \int f\,dx \cdot \int g\,dx$
✓ substitute, by parts, or multiply out.

✗ $\int \cos 3x\,dx = \sin 3x$
✓ $(1/3)\sin 3x$: divide by a.

✗ $\int \sin x\,dx = \cos x$
✓ $-\cos x$.

✗ substituting in a definite integral with the old limits
✓ change them.

✗ area below the axis taken as negative
✓ an area is never negative: split and add sizes.

✗ bottom minus top
✓ top minus bottom.