Unit 18 · Derivatives

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Calculus (Units 17–19 together): 22 of 75 marks, four 1-mark choices, two 5-mark short questions and one 8-mark long question; derivatives appear in all three
- **MCQ (1 mark)**
  - a rule ($\sec x$, e^{x^2}); a stationary point; concave down
  - parametric $x = a\sec\theta$, $y = b\tan\theta$; implicit $xy = 4$; slope at $t = 2$
  - e^{-x} decreasing; least value of a quadratic; three roots of $x^3 - 3x + k = 0$
- **Short (5 marks, in parts)**
  - from the definition; differentiate (product, quotient, chain, inverse, variable power)
  - parametric dy/dx, tangent parallel to an axis; implicit ($x^y = e^{x-y}$)
  - prove a relation in y_1, y_2; monotonicity and extremes of a cubic
  - concavity and inflection; $x > \ln(1 + x)$
- **Long (8 marks)**
  - a cubic: increasing, decreasing, extremes, inflection, greatest and least on an interval
  - a best-value problem: the closed tank of least metal (height = diameter)

Key terms

Average rate of change: $(f(x + h) - f(x))/h$, the slope of a **secant**.

Derivative: its limit as $h \to 0$; the rate of change at an instant; the slope of the **tangent**.

Differentiable: the limit exists. From the definition = **from first principles**.

Critical point: $f'(c) = 0$ or $f'(c)$ doesn't exist; **stationary**: $f'(c) = 0$.

Local maximum / minimum: highest / lowest near c, on both sides.

Concave up: slope increasing, above its tangents; **concave down**: slope decreasing.

Point of inflection: where the concavity changes.

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \qquad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

- written $f'(x)$, dy/dx, y', y_1; d/dx is one symbol, not a fraction
- stone $s = 4.9t^2$: averages 14.7, 10.29, 9.849 → **9.8 m/s** at $t = 1$

Standard derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	x^n	nx^{n-1}
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$	$\csc x$	$-\csc x \cot x$
$\sin^{-1} x$	$1/\sqrt{1-x^2}$, $\lvert x\rvert < 1$	$\cos^{-1} x$	$-1/\sqrt{1-x^2}$
$\tan^{-1} x$	$1/(1+x^2)$	$\cot^{-1} x$	$-1/(1+x^2)$
$\sec^{-1} x$	$1/(\lvert x\rvert\sqrt{x^2-1})$, $\lvert x\rvert > 1$	$\csc^{-1} x$	$-1/(\lvert x\rvert\sqrt{x^2-1})$
e^x	e^x	a^x	$a^x \ln a$
$\ln x$	$1/x$	$\log_a x$	$1/(x \ln a)$

- csc = cosec; radians throughout
- $\sqrt{x} \to 1/(2\sqrt{x})$; $1/x \to -1/x^2$

- $\sin^{-1} x$: defined on $[-1, 1]$, differentiable only on $(-1, 1)$ (vertical tangents at the ends)
- $|x|$ in $\sec^{-1}$: from $\sqrt{x^2}$; without it, right only for $x > 1$
- e^x: its own slope; climbs as steeply as it is high

From the definition

- rewrite $f(x+h) - f(x)$ until h cancels or a standard limit appears
- x^n: $z = x + h \to \lim_{z \to x} (z^n - x^n)/(z - x) = nx^{n-1}$
- **a root**: rationalize; **a fraction**: common denominator
- **sin x**: $\sin(x+h) - \sin x = 2\cos(x + h/2)\sin(h/2)$; $\div h = 2 \cdot h/2$
- **cos x**: $-2\sin(x + h/2)\sin(h/2)$
- **tan x**: top $\sin h$ over $\cos(x+h)\cos x$
- **$\sin^{-1}$x**: $x = \sin y$, $h = \sin(y+k) - \sin y$, $k \to 0 \to 1/\cos y = 1/\sqrt{1 - x^2}$ ($\cos y > 0$)
- **$\tan^{-1}$x**: same steps $\to 1/\sec^2 y = 1/(1 + x^2)$
- **e^x, a^x**: $e^x(e^h - 1)/h$; $a^x(a^h - 1)/h$
- **ln x**: $\ln(1 + h/x)$, $t = h/x \to 1/x$

No derivative at

- a **break** (not continuous)
- a **corner**: $|x|$ at 0, $|h|/h = \pm 1$
- a **vertical tangent**: $\sqrt[3]{x}$ at 0, $h^{-2/3} \to \infty$

Differentiable $\Rightarrow$ **continuous**: $f(a+h) - f(a) = \dfrac{f(a+h) - f(a)}{h} \times h \to f'(a) \times 0$; not the other way ($|x|$)

Rules

$(u \pm v)' = u' \pm v'$, $(cu)' = cu'$	term by term; roots and fractions as powers **Use:** sum, multiple
$(uv)' = uv' + vu'$	add and take away $u(x+h)v(x)$ **Use:** product
$(u/v)' = (vu' - uv')/v^2$	$v \neq 0$; order matters **Use:** quotient
$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$	$\delta y/\delta x = (\delta y/\delta u)(\delta u/\delta x)$ **Use:** chain
$(u^n)' = nu^{n-1}u'$	$(\sin u)' = \cos u \cdot u'$, $(e^u)' = e^u u'$, $(\ln u)' = u'/u$ **Use:** general power

- chain: outside's derivative, inside kept, $\times$ inside's derivative; rates multiply ($3 \times 2 = 6$)
- the others from quotients: cot = cos/sin, sec = 1/cos; $\cos^{-1} x = \pi/2 - \sin^{-1} x$
- an inverse made simple: substitute ($x = \tan\theta$) and use Unit 9's identities
- **variable power** u^v: write $e^{v \ln u}$

Parametric and implicit

- **parametric** $x = f(t)$, $y = g(t)$: $dy/dx = (dy/dt)/(dx/dt)$, $dx/dt \neq 0$; answer in t
  - cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$: no need to remove θ
- **implicit**: differentiate every term in x, y a function of x
  - $(y^2)' = 2yy'$; $(xy)' = xy' + y$; $(\sin y)' = \cos y \cdot y'$
  - collect the y' terms, solve; put in the point

Higher derivatives

- $d^2y/dx^2 = d/dx(dy/dx)$ (y'', y_2); then y_3, ..., y_n
- the rate at which the rate changes; $s \to$ velocity $ds/dt \to$ **acceleration** d^2s/dt^2
- $(e^{ax})^{(n)} = a^n e^{ax}$; $(\sin x)^{(n)} = \sin(x + n\pi/2)$
- an equation in y_1, y_2: clear roots and fractions first, then differentiate
- **parametric**: $d^2y/dx^2 = d/dt(dy/dx) \div dx/dt$, **not** $d^2y/dt^2/d^2x/dt^2$

The slope of the tangent

- $f'(a) = \tan\psi$, ψ the tangent's angle with the positive X-axis ($0 \leq \psi < \pi$)

- $f' > 0$: ψ acute, rising; $f' < 0$: ψ obtuse, falling
- $f' = 0$: tangent **parallel to the X-axis**; $dx/dy = 0$: **parallel to the Y-axis**, no derivative
- parametric: X-axis where $dy/dt = 0$, Y-axis where $dx/dt = 0$
- equations of tangent and normal, rates: Grade 12

Monotonicity

- **increasing**: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$; decreasing: $f(x_1) > f(x_2)$ (some books: *strictly*)
- $f' > 0$ on an interval $\rightarrow$ increasing; $f' < 0 \rightarrow$ decreasing
- $f' = 0$ at single points doesn't spoil it: x^3; $x + \sin x$

1. find $f'(x)$, factorize
2. where $f' = 0$ or doesn't exist: cut the line
3. the sign of f' in each piece
4. sign plus: increasing; minus: decreasing

- an increasing function meets each height once: counts roots, proves inequalities ($x > \ln(1 + x)$), compares e^π and π^e

Extreme values

- local extreme and $f'(c)$ exists $\Rightarrow f'(c) = 0$
  - maximum: $h > 0$ gives quotient ≤ 0, $h < 0$ gives ≥ 0
- every extreme at a critical point; not every critical point an extreme (x^3 at 0)

First derivative test (sign of f' left $\rightarrow$ right of c)

- from plus to minus: **maximum**; minus to plus: **minimum**; no change: none
- works at corners: $|x|$ has a minimum at 0

Second derivative test ($f'(c) = 0$)

- $f''(c) < 0$: **maximum**; $f''(c) > 0$: **minimum**
- $f''(c) = 0$: no conclusion $\rightarrow$ first derivative test

On [a, b]: compare f at the critical points inside **and at both ends**

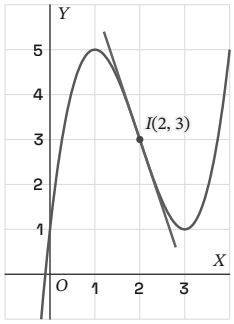
Best-value problems: the quantity as a function of one variable (the condition removes the other), then its extreme

- closed tank of volume V: $S = 2\pi r^2 + 2\pi rh$, $h = V/(\pi r^2) \rightarrow$ least when $h = 2r$

The Pelton wheel: jet v, cups u

- water arrives at $v - u$, turned back: change $2(v - u)$
- $P(u) = 2\rho Q(v - u)u$; $P' = 2\rho Q(v - 2u) = 0 \rightarrow u = v/2$; $P'' = -4\rho Q < 0$: maximum
- water leaves at $v/2$ backwards from a cup moving $v/2$ forwards: it drops

Concavity and inflection



- $f'' > 0$: concave up (cup)
- $f'' < 0$: concave down (cap)
- inflection where f'' changes sign

- inflection: $f''(c) = 0$ if it exists, but not enough: x^4 at 0 is concave up both sides
- find f'', its zeros, its sign on each piece; keep only sign changes
- at an inflection f' is greatest or least nearby: the fastest rise or fall
- every cubic $ax^3 + bx^2 + cx + d$: one inflection at $x = -b/(3a)$

COMMON MISTAKES

✗ $(uv)' = u'v'$
✓ $uv' + vu'$.

✗ quotient $(uv' - vu')/v^2$
✓ $(vu' - uv')/v^2$: order matters.

✗ $(\sin x^2)' = \cos x^2$
✓ $2x\cos x^2$: chain rule.

✗ $(a^x)' = xa^{x-1}$
✓ $a^x \ln a$; the power rule needs a constant power.

✗ $f'(c) = 0$, so an extreme
✓ test it: x^3 at 0.

✗ $f''(c) = 0$, so an inflection
✓ f'' must change sign.

✗ $d^2y/dx^2 = (d^2y/dt^2)/(d^2x/dt^2)$
✓ differentiate dy/dx in t, $\div\, dx/dt$.

✗ greatest value on [a, b] from the critical points only
✓ check the ends too.

✗ derivatives in degrees
✓ radians.