Unit 17 · Limits and Continuity

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Calculus (Units 17–19 together): 22 of 75 marks, four 1-mark choices, two 5-mark short questions and one 8-mark long question
- limits and continuity usually bring a choice (a standard limit, a type of discontinuity) and a short question: two or three limits, or a function in pieces tested for continuity, a constant found
- **MCQ (1 mark)**
  - the condition for continuity; which form is indeterminate
  - $\sin 7x/x$; a quotient at infinity; $(x^4-1)/(x-1)$; $(e^{2x}-1)/x$
  - the discontinuity of $1/(x-2)$; k for continuity; $\sin x^\circ/x$
- **Short (5 marks, in parts)**
  - factorize, rationalize, $\sqrt{x^2+3x}-x$ at infinity
  - $(\sin 2x+\sin 6x)/x$; $(\sin x-\sin a)/(x-a)$
  - $(e^{ax}-e^{bx})/x$; $(1+3x)^{2/x}$; $\ln(1+4x)/(e^{2x}-1)$
  - continuity of a function in pieces, the type, the mend; k; $|x|$ at 0
  - points of discontinuity of a rational function; a root between 0 and 1
- **Long (8 marks)**
  - LHL and RHL at two joins; continuity; the types; the sketch
  - a limit at $\pi/4$; a, b for continuity

Key terms

Limit $\lim_{x\to a} f(x)=l$: $f(x)$ closes in on l as x closes in on a from either side, **never equal to** a.

LHL, **RHL**: the limits from the left ($x<a$) and from the right ($x>a$).

Indeterminate form: a form that doesn't decide the limit.

Continuous at a: $\lim_{x\to a} f(x)=f(a)$.

Point of discontinuity: where any of the three conditions fails.

Greatest integer $[x]$: the greatest whole number not more than x; $[-1.5]=-2$.

Formulas

$\lim_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$	rational n; $a>0$ if not who **Use:** algebraic
$\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1,\ \dfrac{\tan\theta}{\theta}\to 1$	θ in **radians** **Use:** trigonometric
$\dfrac{\sin kx}{x}\to k;\ \dfrac{1-\cos\theta}{\theta^2}\to\dfrac{1}{2}$	as $x,\theta\to 0$ **Use:** from $\sin\theta/\theta$
$\lim_{x\to 0}(1+x)^{1/x}=e$	$=\lim_{n\to\infty}(1+1/n)^n=2.7$ **Use:** the number e
$\dfrac{e^x-1}{x}\to 1;\ \dfrac{a^x-1}{x}\to\ln a$	as $x\to 0$ **Use:** exponential
$\dfrac{\ln(1+x)}{x}\to 1;\ \dfrac{\log_a(1+x)}{x}\to\log_a e$	as $x\to 0$ **Use:** logarithmic
$(1+k/x)^x\to e^k$	as $x\to\infty$ **Use:** at infinity

Limit of a function

- $(x^3-1)/(x-1)$: no value at 1, but 2.997… and 3.003… on either side → limit 3

- the limit says nothing about $f(a)$: it may be the limit, something else, or undefined
- LHL $= \lim_{h \to 0} f(a-h)$, RHL $= \lim_{h \to 0} f(a+h)$, $h > 0$
- **the limit exists** $\Leftrightarrow$ **LHL = RHL** (both finite)
- **exactly**: for every $\varepsilon > 0$ there is a $\delta > 0$ with $|f(x) - l| < \varepsilon$ whenever $0 < |x - a| < \delta$
  - $\lim_{x \to 3}(2x+1) = 7$: $\delta = \varepsilon/2$
  - not asked in NEB's exam

Indeterminate forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad 0 \times \infty, \quad 1^\infty, \quad 0^0, \quad \infty^0$$

- $x^2/x \to 0$, $2x/x \to 2$, $x/x^2 \to \infty$: same 0/0, three answers
- a signal to rewrite (factorize, rationalize, a standard limit), **never an answer**
- **not** indeterminate: $\infty + \infty$, $\infty \times \infty$ (unbounded); $0/\infty$, 0^∞ (0)
- ∞ is not a number: "$\to \infty$" means larger than any number named
- L'Hospital's rule: Grade 12

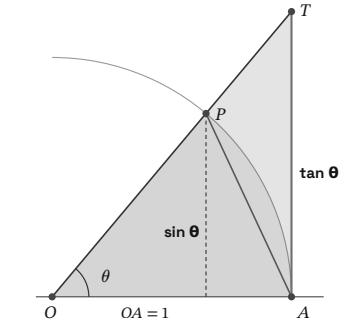
Properties

- limit of sum, difference, $k\times$, product, quotient (bottom's limit $\neq 0$), power, root = that of the limits
- **only when each limit exists**: $\lim(1/x - 1/x) = 0$, but neither $\lim 1/x$ exists
- polynomial, or rational with bottom $\neq 0$ at a: **put $x = a$ in**

Algebraic limits

- $0/0 \to$ a hidden factor $x - a$: factorize or rationalize, **cancel** (allowed: $x \neq a$), put $x = a$
- $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \ldots + a^{n-1})$: n terms, each $\to a^{n-1}$
- negative n, fractions: the same result
- it is Unit 18's power rule
- **at infinity**: divide top and bottom by the bottom's highest power; $1/x^n \to 0$ $(n > 0)$
- $\infty - \infty$ with roots: multiply by the conjugate

Trigonometric limits



- $\triangle OAP \subset$ sector $OAP \subset \triangle OAT$, radius 1
- ½ sin θ < ½ θ < ½ tan θ

1. $\div$ ½ sin θ: $1 < \theta/\sin\theta < 1/\cos\theta$
2. turn over: $\cos\theta < \sin\theta/\theta < 1$
3. $\cos\theta \to 1$: squeezed to 1 (**sandwich theorem**)
4. $\theta < 0$: $\sin(-\theta)/(-\theta) = \sin\theta/\theta$: LHL = RHL

- **radians** only: in degrees, $\sin x^\circ/x \to \pi/180$
- $(\tan\theta)/\theta = (\sin\theta)/\theta \cdot 1/(\cos\theta)$
- $(1 - \cos\theta)/\theta^2 = (2\sin^2(\theta/2))/\theta^2 \to 1/2$
- make every sine look like $\sin\theta/\theta$ with the **same** θ
- a limit at $a \neq 0$: put $y = x - a$

Exponential and logarithmic limits

n	1	10	100	1000	1 000 000
$(1 + 1/n)^n$	2	2.5937	2.7048	2.7169	2.71828

- e irrational; $\ln x = \log_e x$, natural logarithm
- **$\ln(1+x)/x$**: $= \ln(1+x)^{1/x} \to \ln e = 1$
- **$(e^x - 1)/x$**: $t = e^x - 1$, $x = \ln(1+t) \to 1/(\ln(1+t)/t) \to 1$
- **$(a^x - 1)/x$**: $a^x = e^{x \ln a} \to \ln a$
- **$\log_a(1+x)/x$**: change of base, $\div \ln a$
- $(1 + k/x)^x = [(1 + k/x)^{x/k}]^k \to e^k$
- only $y = e^x$ crosses the Y-axis at slope exactly 1
- make the same small quantity appear in the power (or log) and below it

The bank: $P(1 + r/n)^n \to Pe^r$

- Rs 1,00,000 at 10%: yearly 1,10,000; monthly 1,10,471.31; daily 1,10,515.58
- every instant: $100000\,e^{0.1} \approx$ Rs 1,10,517.09, never more: **continuous compounding**

Continuity

LHL = RHL = f(a)

1. $f(a)$ defined

2. the limit exists
3. the two equal

- or $\lim_{h \to 0} f(a+h) = f(a)$: a small change in x, a small change in $f(x)$
- on (a, b): at every point; on $[a, b]$: also from the right at a, from the left at b
- sums, products, quotients (bottom $\neq 0$), compositions of continuous functions: continuous
- everywhere: polynomials, sin, cos, e^x, a^x, $|x|$
- rational: except where the bottom is 0; tan, sec: except odd multiples of $\pi/2$
- $\ln x$: $x > 0$; $\sqrt{x}$: $x \geq 0$
- check where the formula fails and **at each join** of a function in pieces
- a constant: make LHL = RHL = $f(a)$
- **intermediate value theorem**: continuous on $[a, b] \to$ takes every value between $f(a)$ and $f(b)$
  - opposite signs $\to$ a root between; $x^3 - 4x + 1$: $f(0) = 1, f(1) = -2$

Types of discontinuity

Kind	Type	One-sided limits	Example	Removable?
First	removable	equal, $\neq f(a)$ or $f(a)$ undefined	$(x^2 - 1)/(x - 1)$ at 1	yes, redefine $f(a)$
First	jump	finite, unequal; jump = RHL − LHL	$[x]$ at a whole number; $\lvert x-2\rvert/(x-2)$	no
Second	infinite	at least one $\pm\infty$ (an asymptote)	$1/x$, $1/x^2$ at 0; tan x at $\pi/2$	no
Second	oscillating	at least one doesn't exist	$\sin(1/x)$ at 0	no

- some books call only the jump "first kind": name the type and you are right either way

COMMON MISTAKES

✗ 0/0 is the answer (or 1)
✓ a signal to rewrite.

✗ $\lim_{x \to a} f(x) = f(a)$ always
✓ only when f is continuous there.

✗ limit exists when one side is found
✓ need LHL = RHL.

✗ $\sin x/x \to 1$ in degrees
✓ $\pi/180$; use radians.

✗ $(\sin 3x)/(\sin 5x) \to 1$
✓ 3/5: make each look like $\sin\theta/\theta$.

✗ $1^\infty = 1$
✓ indeterminate; $(1 + 1/n)^n \to e$.

✗ splitting a limit into limits that don't exist
✓ check each exists first.

✗ $1/x^2$ at 0 continuous because both sides go to $+\infty$
✓ ∞ is not a value.