Unit 14 · Vectors

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Vectors: 6 of 75 marks; with trigonometry and analytic geometry they set three 1-mark choices (at least one on vectors), two 5-mark short questions and one 8-mark long question
- **CDC's model paper:** the condition that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ gives independent vectors (1 mark); when $(1+a)\vec{i} + b\vec{j} + c\vec{k}, a\vec{i} + (1+b)\vec{j} + c\vec{k}, a\vec{i} + b\vec{j} + (1+c)\vec{k}$ are dependent (5 marks)
- **MCQ (1 mark)**
  - a set with $\vec{0}$; four vectors in space; the name "linear combination"
  - $\vec{i} + \vec{j}, \vec{j} + \vec{k}, \vec{k} + \vec{i}$; collinear points ($p$); coplanar ($\lambda$)
  - independence of two plane vectors; (5, 1) from (1, 1), (1, −1)
- **Short (5 marks, in parts)**
  - three collinear points, the ratio; four coplanar points
  - independent, then $\vec{r}$ as their combination; the values making three dependent
  - diagonals of a parallelogram bisect; balancing an equation; collinear in terms of $\vec{a}, \vec{b}, \vec{c}$
- **Long (8 marks)**
  - independent; $\vec{d}$ as a combination; are the four dependent?
  - $x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x = y = 0$; the medians meet at 2 : 1

Key terms

Collinear (parallel) vectors: parallel to one line; one is a number times the other, **like** if the number is positive, **unlike** if negative.

Coplanar vectors: parallel to one plane. Any two are; three need not be.

Linear combination: numbers times the vectors, added.

Linearly dependent: a combination equal to the zero vector with the numbers **not all zero**.

Linearly independent: that happens **only** when every number is 0.

Basis: two non-collinear vectors in a plane, three non-coplanar in space.

- collinear: $\vec{b} = \lambda\vec{a}$; $\lambda > 0$ like, $\lambda < 0$ unlike
- combination $x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n$; dependent when it is $\vec{0}$ with the xs not all zero
- $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = (a_1, a_2, a_3)$; $|\vec{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$
- $\overrightarrow{AB} = \vec{b} - \vec{a}$ (end minus start); vectors are free; $\vec{0}$ has no direction

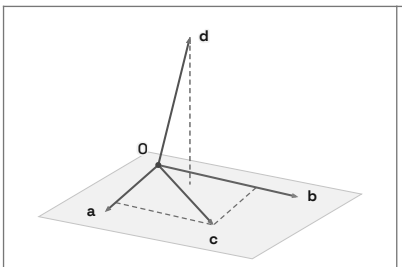
Collinear

- components **proportional**: one λ for all three
- (2, −3, 1), (−4, 6, −2): $\lambda = -2$, collinear; (4, −6, 1): not
- test with one λ, not $b_1/a_1 = b_2/a_2 = \ldots$: (0, 2, 1), (0, 4, 2) collinear though 0/0
- $\vec{0} = 0 \cdot \vec{a}$: collinear with every vector
- **points**: A, B, C on a line $\Leftrightarrow \overrightarrow{AC} = t\overrightarrow{AB} \Leftrightarrow \vec{c} = (1-t)\vec{a} + t\vec{b}$ (numbers add to 1)

Coplanar

$\vec{a}, \vec{b}$ non-collinear: $\vec{a}, \vec{b}, \vec{c}$ coplanar $\Leftrightarrow \vec{c} = x\vec{a} + y\vec{b}$

1. if $\vec{c} = x\vec{a} + y\vec{b}$: the parallelogram's diagonal, in their plane
2. if $\vec{c}$ is in the plane: through C a line parallel to $\vec{b}$ meets $\vec{a}$'s line at M; $\overrightarrow{OM} = x\vec{a}$, $\overrightarrow{MC} = y\vec{b}$



$\vec{c} = x\vec{a} + y\vec{b}$ in the plane of $\vec{a}$ and $\vec{b}$

- **four points** coplanar $\Leftrightarrow \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ coplanar

Linear combination

- **Theorem 1 (plane)**: $\vec{a}, \vec{b}$ non-collinear $\rightarrow$ every $\vec{r}$ in their plane is $x\vec{a} + y\vec{b}$, **one way only**
  - two ways: $(x - x')\vec{a} = (y' - y)\vec{b}$; $x \neq x'$ makes $\vec{a}, \vec{b}$ collinear
- **Theorem 2 (space)**: $\vec{a}, \vec{b}, \vec{c}$ non-coplanar $\rightarrow$ every $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$, one way only
  - through R parallel to $\vec{c}$, meet the plane of $\vec{a}, \vec{b}$ at M: $\overrightarrow{OM} = x\vec{a} + y\vec{b}$, $\overrightarrow{MR} = z\vec{c}$
- so **compare coefficients**: $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \rightarrow x = x', y = y'$
- to find x, y, z: equate components $\rightarrow$ simultaneous equations

Dependent and independent

- dependent $\Leftrightarrow$ one is a combination of the others (divide by a non-zero x)
- **all zero** always works: it proves nothing

In pictures

1. any set with $\vec{0}$: dependent $(1 \cdot \vec{0} = \vec{0})$
2. **two**: dependent $\Leftrightarrow$ collinear
3. **three**: dependent $\Leftrightarrow$ coplanar
4. **four in space** (three in a plane): always dependent

The determinant test

$$\vec{a}, \vec{b}, \vec{c} \text{ dependent} \Leftrightarrow \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 0$$

- $\Rightarrow$: $\vec{c} = p\vec{a} + q\vec{b}$; $R_3 \rightarrow R_3 - pR_1 - qR_2$ makes a zero row
- $\Leftarrow$: $(\text{adj } A)A = |A|I = O$: a non-zero row of adj A gives the dependence; all cofactors $0 \rightarrow \vec{a}, \vec{b}$ collinear
- plane: $a_1\vec{i} + a_2\vec{j}, b_1\vec{i} + b_2\vec{j}$ dependent $\Leftrightarrow a_1b_2 - a_2b_1 = 0$
- non-zero determinant $\rightarrow$ independent
- the relation: solve $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ by components
- given as $p\vec{a} + q\vec{b} + r\vec{c}$ with $\vec{a}, \vec{b}, \vec{c}$ non-coplanar: the test works on the numbers (p, q, r)

Photosynthesis: (C, H, O) counts

- $CO_2 = (1, 0, 2)$, $H_2O = (0, 2, 1)$, glucose $(6, 12, 6)$, $O_2 = (0, 0, 2)$
- four vectors in a three-number space $\rightarrow$ always dependent $\rightarrow$ a balance exists
- C: $x = 6z$; H: $y = 6z$; O: $w = 6z \rightarrow 6, 6, 1, 6$
- CO_2, H_2O, O_2 independent (determinant 4) $\rightarrow$ only one balance, up to a multiple
- a negative number: that molecule on the other side; more elements than molecules: maybe nothing balances

For the engineering entrance (IOE)

- **vector product** $\vec{b} \times \vec{c}$: perpendicular to both; length the parallelogram's area
- **scalar triple product** $[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$ = the determinant of their components
- size = **volume** of the parallelepiped on them; tetrahedron: 1/6 of it
- coplanar $\Leftrightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0$
- cyclic order keeps the value; swapping two changes its sign: $[\vec{i}\,\vec{k}\,\vec{j}] = -1$

COMMON MISTAKES

✗ all-zero numbers give the zero vector, so independent
✓ that is true of every set; independence needs it to be the **only** way.

✗ collinear tested by $b_1/a_1 = b_2/a_2 = b_3/a_3$ with a 0
✓ find one λ.

✗ AB as start minus end
✓ end minus start: B's vector minus A's.

✗ three vectors in space always dependent
✓ only when coplanar; **four** always.

✗ determinant non-zero, so dependent
✓ non-zero means independent.

✗ comparing coefficients of two collinear vectors
✓ only for non-collinear (non-coplanar) vectors.