Unit 13 · Coordinates in Space

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Analytic geometry (Units 11–13 together): 9 of 75 marks; with trigonometry and vectors it sets three 1-mark choices, two 5-mark short questions and one 8-mark long question
- **CDC's model paper:** the direction cosines of a line whose direction cosines are proportional to 1, −3, 2 (3 marks)
- **MCQ (1 mark)**
  - where a point lies; distance from the origin or from a coordinate plane
  - direction cosines of an axis; which numbers can't be direction cosines
  - from direction ratios; equal angles with the axes; $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$
  - the foot on a coordinate plane; k for collinear points
- **Short (5 marks, in parts)**
  - a distance; the point on an axis equally far from two points
  - direction cosines from ratios, from two points; the angles with the axes
  - three points make an equilateral triangle
- **Long (8 marks)**
  - a right-angled isosceles triangle; direction cosines; area; the fourth corner of a square

Key terms

Axes $X'OX$, $Y'OY$, $Z'OZ$: each at right angles to the other two, meeting at the **origin** O.

Right-handed: right hand's fingers curl from OX to OY, thumb along OZ.

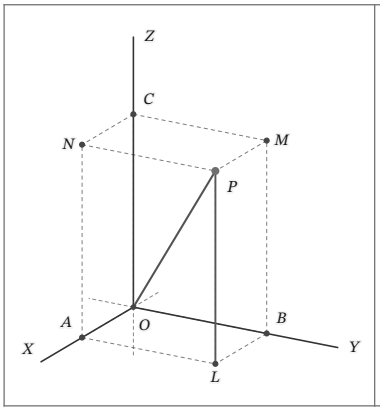
Coordinate planes: XY, YZ, ZX; they cut space into eight **octants**.

Coordinates (x, y, z): signed distances of P from the YZ-, ZX- and XY-planes.

Direction cosines l, m, n: $\cos\alpha, \cos\beta, \cos\gamma$, the angles (0°–180°) with the positive axes.

Direction ratios a, b, c: any three numbers proportional to l, m, n.

Points



- $P(x, y, z)$: the box with O and P at opposite corners
- $x = OA$, $y = OB$, $z = OC$

- on the X-axis $(x, 0, 0)$; in the XY-plane $(x, y, 0)$: a coordinate is 0 when the point is in the plane of the other two axes

Octant	I	II	III	IV	V	VI	VII	VIII
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

- octants are numbered differently in different books: the signs are what count

Distance

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

1. $S(x_2, y_1, z_1)$, $R(x_2, y_2, z_1)$: $PR^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ (Pythagoras in PSR)
2. $RQ = |z_2 - z_1|$, perpendicular to PR
3. $PQ^2 = PR^2 + RQ^2$ (Pythagoras again)

- $OP = \sqrt{x^2 + y^2 + z^2}$
- right angle: $a^2 + b^2 = c^2$; isosceles: two sides equal; collinear: $AB + BC = AC$
- a point on the Z-axis: $(0, 0, z)$; equally far: $PA^2 = PB^2$

Direction cosines and ratios

$$l^2 + m^2 + n^2 = 1$$

1. $P(x, y, z)$ on the line through O, $OP = r$; foot A on the X-axis, right angle at A
2. $x = r\cos\alpha = lr$; $y = mr$; $z = nr$
3. $x^2 + y^2 + z^2 = r^2 \to (l^2 + m^2 + n^2)r^2 = r^2$

- with $r = 1$: l, m, n are the coordinates of the point 1 unit along the line
- reversed line: $-l$, $-m$, $-n$
- axes: X (1, 0, 0), Y (0, 1, 0), Z (0, 0, 1)
- two angles fix the third, up to a sign
- equal angles with all three axes: $l = m = n = \pm 1/\sqrt{3}$

From ratios to cosines: $a^2 + b^2 + c^2 = k^2$

$$l = \pm\frac{a}{k}, \qquad m = \pm\frac{b}{k}, \qquad n = \pm\frac{c}{k}$$

- the **same sign** all through (one for each way along the line)
- **through two points**: ratios $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$; divide by PQ for the cosines
- proportional ratios and a common point → **collinear**
- angles with the axes: $\alpha = \cos^{-1} l$, ...

Kala Patthar to Everest: origin on Kala Patthar, X east, Y north, Z up; summit $Q(9.51, -0.83, 3.20)$ km

- $OQ \approx 10.07$ km
- $l = 0.944$, $m = -0.082$, $n = 0.318$
- $\gamma \approx 71.5°$ from the vertical → look up **18.5°**

For the engineering entrance (IOE)

- **angle between two lines**: $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$ (cosine rule in OPQ, $OP = OQ = 1$)
- with ratios:

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}\sqrt{a_2{}^2 + b_2{}^2 + c_2{}^2}}$$

- **perpendicular**: $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$; **parallel**: $a_1/a_2 = b_1/b_2 = c_1/c_2$
- **section formula** $k_1 : k_2$: $(k_1 x_2 + k_2 x_1)/(k_1 + k_2)$, and so for y, z; midpoint the average
- **plane** through (x_1, y_1, z_1) with **normal** a, b, c: $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$; every plane $ax + by + cz + d = 0$
- intercept form: $x/p + y/q + z/r = 1$
- distance from a plane: $|ax_1 + by_1 + cz_1 + d|/\sqrt{a^2 + b^2 + c^2}$
- angle between planes = angle between normals; perpendicular when $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

COMMON MISTAKES

✗ direction ratios given as the cosines
✓ divide by $\sqrt{a^2 + b^2 + c^2}$.

✗ different signs on l, m, n
✓ one sign all through.

✗ $l + m + n = 1$
✓ $l^2 + m^2 + n^2 = 1$.

✗ distance from the XY-plane is $\sqrt{x^2 + y^2}$
✓ $|z|$.

✗ angle with the Z-axis taken as the elevation
✓ elevation $= 90° - \gamma$.

✗ the plane distance formula without the size bars
✓ always $|\ldots|$.