Unit 12 · Pair of Straight Lines

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Analytic geometry (Units 11–13 together): 9 of 75 marks; with trigonometry and vectors it shares three 1-mark choices, two 5-mark short questions and one 8-mark long question
- **CDC's model paper:** in a long question, the one second-degree equation of two given lines (2 marks), then a bisector (3 marks)
- **MCQ (1 mark)**
  - perpendicular ($a + b = 0$); coincident ($h^2 = ab$)
  - the lines of $x^2 - 7xy + 12y^2 = 0$; the acute angle; product of the slopes
  - an imaginary pair; the bisectors; k for a pair
  - one slope twice the other; two pairs bisecting each other; where the lines meet
- **Short (5 marks, in parts)**
  - k for a pair of lines; the acute angle
  - two lines as one equation; the parallel pair through O and its bisectors
  - the perpendicular pair is $bx^2 - 2hxy + ay^2 = 0$
  - show an equation is a pair; find the lines and the angle
- **Long (8 marks)**
  - prove the bisectors $h(x^2 - y^2) = (a - b)xy$, perpendicular; use them; check with Unit 11's formula

Key terms

General second-degree equation: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$; h, g, f are **halves** of the coefficients of xy, x, y.

Pair of lines: the product of two first-degree expressions $= 0$: on one line **or** the other.

Homogeneous: all terms of the same degree: $ax^2 + 2hxy + by^2 = 0$.

Coincident: one line counted twice.

- $(2x - y + 1)(x + 2y - 3) = 0 \Leftrightarrow 2x^2 + 3xy - 2y^2 - 5x + 5y - 3 = 0$: $h = 3/2$, $g = -5/2, f = 5/2$
- the converse fails: $x^2 + y^2 - 4 = 0$ is a circle

Formulas

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$	the determinant of $a, h, g; h, b, f; g, f, c$ **Use:** general equation is a pair
$m_1 + m_2 = -2h/b,\ m_1 m_2 = a/b$	slopes: roots of $bm^2 + 2hm + a = 0$ **Use:** homogeneous pair
$\tan\theta = \dfrac{2\sqrt{h^2 - ab}}{\lvert a + b\rvert}$	acute angle; general pair too **Use:** angle between the lines
$h(x^2 - y^2) = (a - b)xy$	or $\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$ **Use:** bisectors

Condition for a pair

1. as a quadratic in x: $ax^2 + 2(hy + g)x + (by^2 + 2fy + c) = 0$
2. two lines only if the quantity under the root, $(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)$, is a perfect square

3. $Py^2 + 2Qy + R$ a square $\Leftrightarrow Q^2 = PR$: $(gh - af)^2 = (h^2 - ab)(g^2 - ac)$
4. expand, cancel g^2h^2, divide by $-a$

$$\det \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} = 0$$

- real lines need $h^2 \geq ab$; crossing: $h^2 > ab$
- $x^2 + y^2 = 0$ passes the test but $h^2 - ab = -1$: imaginary pair, only (0, 0)
- $x^2 + 1 = 0$: imaginary parallel pair, no real point
- $h^2 = ab$: parallel (general pair)
- to find the lines: solve as a quadratic in x (or y): the root comes out exact

Homogeneous pair: through the origin

- (x, y) fits $\rightarrow$ (tx, ty) fits: a whole line through O
- $y = mx$ on it: $bm^2 + 2hm + a = 0$; $ax^2 + 2hxy + by^2 = b(y - m_1x)(y - m_2x)$
- $h^2 > ab$: real, different; $h^2 = ab$: coincident; $h^2 < ab$: only O
- $b = 0$: $x(ax + 2hy) = 0$, one line the Y-axis
- perpendicular pair through O: $bx^2 - 2hxy + ay^2 = 0$

Angle

1. $\tan\theta = |(m_1 - m_2)/(1 + m_1m_2)|$
2. $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2 = 4(h^2 - ab)/b^2$
3. $1 + m_1m_2 = (a + b)/b$; divide

- **perpendicular**: $a + b = 0$ (coefficients of x^2, y^2 add to 0)
- **coincident**: $h^2 = ab$ ($\theta = 0$)
- general pair: its second-degree terms $ax^2 + 2hxy + by^2 = 0$ are the parallel pair through O: same angle

Bisectors

1. Unit 11: $(y - m_1x)/\sqrt{1 + m_1^2} = \pm(y - m_2x)/\sqrt{1 + m_2^2}$
2. square (both signs at once)
3. expand, divide by $m_1 - m_2$: $(m_1 + m_2)(x^2 - y^2) = 2(1 - m_1m_2)xy$
4. put in the sum and product; $\times -b/2$

- coefficients of x^2, y^2: h, $-h$ $\rightarrow$ bisectors **perpendicular**
- $a = b$: $y = \pm x$; $h = 0$: the axes (use the first form)

The drum: delay $\rightarrow$ path difference 12.9 cm of 21.5 $\rightarrow$ $\cos\varphi = 3/5$

- $x/\sqrt{x^2 + y^2} = 3/5 \rightarrow 16x^2 - 9y^2 = 0 \rightarrow y = \pm(4/3)x$, 53.1° from the ears' line
- bisectors $xy = 0$: the two directions are mirror images in the ears' line: front and back can't be told

For the engineering entrance (IOE)

- **where the lines meet**: solve $ax + hy + g = 0$, $hx + by + f = 0$
- **bisectors of a general pair**: from the meeting point (x_0, y_0): $h[(x - x_0)^2 - (y - y_0)^2] = (a - b)(x - x_0)(y - y_0)$
- **origin to where a line cuts a curve**: write the line as $(\ldots) = 1$; multiply the curve's terms by its powers until every term has degree 2 $\rightarrow$ a homogeneous pair

COMMON MISTAKES

✗ $h = 3$ from $3xy$
✓ $h = 3/2$: h, g, f are halves.

✗ determinant 0, so two real lines
✓ also need $h^2 \geq ab$.

✗ perpendicular when $h = 0$
✓ when $a + b = 0$.

✗ $m_1m_2 = b/a$
✓ a/b; $m_1 + m_2 = -2h/b$.

✗ bisectors $h(x^2 - y^2) = (a + b)xy$
✓ $(a - b)xy$.

✗ squaring kept both halves of each line
✓ check which half fits (the drum's side).