Unit 10 · Trigonometric Equations and General Values

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Trigonometry (Units 9 and 10 together): 6 of 75 marks, usually a 1-mark choice and a 5-mark short question; can join the 8-mark long question with analytic geometry and vectors
- **CDC's model paper:** general solutions of $\sin x = 1$, $\cos x = 0$, $\tan x = 0$ (3 marks); a choice on the quadrant of the solutions of $a\cos\theta + b\sin\theta = c$
- **MCQ (1 mark)**
  - a general solution ($\sin\theta = 0$, $\cos\theta = -1/2$, $\tan^2\theta = 3$)
  - two equations together; an equation with **no** solution
  - solutions in an interval; for which k a solution exists
- **Short (5 marks, in parts)**
  - general value from sec or cosec
  - a quadratic in one ratio; $\sin 5x = \sin 3x$ two ways
  - $\cos x - \sin x = 1/\sqrt{2}$; homogeneous; solutions in $[0, 2\pi]$
- **Long (8 marks)**
  - prove $\theta = 2n\pi \pm \alpha$; a sum of cosines; solutions in an interval
  - an a.c. voltage: instants, how many a second, a fraction of each cycle

Key terms

Identity: true for **every** angle ($\sin^2\theta + \cos^2\theta = 1$).

Trigonometric equation: the unknown angle inside a ratio; true only for some angles.

Solutions in an interval: those in a named range (in $[0, 2\pi)$: some books' *principal solutions*).

General solution (general value): one formula with $n \in \mathbb{Z}$ giving every solution.

Homogeneous: every term of the same degree in sin and cos.

- ratios repeat → one solution means infinitely many

Ratio 0 or ±1 (from the unit circle)

Equation	General solution
$\sin\theta = 0$, $\tan\theta = 0$	$\theta = n\pi$
$\cos\theta = 0$	$\theta = (2n+1)\pi/2$
$\sin\theta = 1$	$\theta = 2n\pi + \pi/2 = (4n+1)\pi/2$
$\sin\theta = -1$	$\theta = (4n-1)\pi/2$
$\cos\theta = 1$	$\theta = 2n\pi$
$\cos\theta = -1$	$\theta = (2n+1)\pi$

- ± 1: one point of the circle → a whole turn apart; 0: two opposite points → half a turn
- $\cot\theta = 0$ is $\cos\theta = 0$; sec and cosec are never 0

General values

$\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n\alpha$	same for cosec **Use:** α and $\pi - \alpha$
$\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha$	same for sec **Use:** α and $-\alpha$
$\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$	same for cot **Use:** half-turn repeat
$\sin^2\theta = \sin^2\alpha$ (or $\cos^2$, $\tan^2$) $\Leftrightarrow \theta = n\pi \pm \alpha$	via $\cos 2\theta = \cos 2\alpha$ **Use:** squares

$\sin\theta = \sin\alpha$, proof

1. $\sin\theta - \sin\alpha = 2\cos((\theta+\alpha)/2)\sin((\theta-\alpha)/2) = 0$
2. $\sin((\theta-\alpha)/2) = 0 \to \theta = 2m\pi + \alpha$ (even multiple, plus)
3. $\cos((\theta+\alpha)/2) = 0 \to \theta = (2m+1)\pi - \alpha$ (odd multiple, minus)
4. $(-1)^n$ joins both

- cos: $-2\sin((\theta+\alpha)/2)\sin((\theta-\alpha)/2) = 0 \to 2n\pi \pm \alpha$
- tan: $\sin(\theta-\alpha) = 0 \to n\pi + \alpha$
- a square hides a sign: $\sin^2\theta = 1/4$ is $\sin\theta = \pm 1/2$
- α: the principal value (Unit 9); not a standard value $\to$ leave it: $n\pi + \tan^{-1} 3$
- degrees: $180°n + (-1)^n\, 30°$
- **multiple angle**: solve for $k\theta$, then divide by k
- **check**: put $n = 0, 1$ back into the equation

Solving: the method

Change the equation into $\sin\theta = k$, $\cos\theta = k$, $\tan\theta = k$; write each general value.

- **one ratio**: $\sin^2 + \cos^2 = 1$, then a quadratic; a root outside $[-1, 1]$ is thrown away
  - $2\sin^2\theta + 3\cos\theta = 0 \to$ quadratic in $\cos\theta$
- **factorize**; a sum $\to$ a product ($\cos x + \cos 5x$): each factor its own family
- **never divide** by a factor that can be 0: in $\sin 2x = \cos x$, dividing by $\cos x$ loses $(2n+1)\pi/2$
- **homogeneous**: check $\cos x = 0$, then $\div \cos^2 x \to$ quadratic in $\tan x$
- **in an interval**: general solution, then $n = 0, \pm 1, \pm 2, \ldots$, keep those inside

$a\cos\theta + b\sin\theta = c$

1. $r = \sqrt{a^2+b^2}$; divide by r
2. $\cos\beta = a/r$, $\sin\beta = b/r$
3. left side $= \cos(\theta - \beta)$
4. $\cos(\theta-\beta) = c/r$: solutions **only if** $|c| \le r$

- or $r\sin(\theta+\gamma)$, $\sin\gamma = a/r$, $\cos\gamma = b/r$: whichever gives a standard angle
- $a\cos\theta + b\sin\theta$ takes every value from $-r$ to r, no other
- **squaring** adds solutions: $\sin\theta + \cos\theta = 1$ squared $\to n\pi/2$, but π and $3\pi/2$ fail; true answer $2n\pi$ or $2n\pi + \pi/2$
- multiplying by $\cos\theta$ (to clear tan, sec) can let in $\cos\theta = 0$: check every family

The socket: $230\sqrt{2}\sin(100\pi t) = 230$

1. $\sin(100\pi t) = 1/\sqrt{2} = \sin(\pi/4)$
2. $100\pi t = n\pi + (-1)^n\, \pi/4$
3. $t = n/100 + (-1)^n/400$ s

- 2.5 ms, 7.5 ms, 22.5 ms, 27.5 ms; cycle 20 ms
- +230 V twice a cycle: 100 times a second; ±230 V: 200 times
- above 230 V a quarter of each cycle, below −230 V another quarter

COMMON MISTAKES

✗ $\sin\theta = 1/2 \to 2n\pi \pm \pi/6$
✓ that is cosine's pattern; $n\pi + (-1)^n\, \pi/6$.

✗ one solution given
✓ the general solution, with $n \in \mathbb{Z}$.

✗ $\cos x = 2$ solved
✓ no angle: throw away $|k| > 1$.

✗ dividing by $\cos x$
✓ take it out as a factor.

✗ solutions after squaring kept
✓ check every family in the original.

✗ $2\theta = \ldots$ then forgetting to divide
✓ $\theta =$ half of it.

✗ $a\cos\theta + b\sin\theta = c$ always solvable
✓ only if $|c| \le \sqrt{a^2+b^2}$.