Unit 9 · Inverse Circular Functions

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Trigonometry (Units 9 and 10 together): 6 of 75 marks, usually a 1-mark choice and a 5-mark short question
- **CDC's model paper:** the domain and range of $y = \sin^{-1} x$ (2 marks)
- **MCQ (1 mark)**
  - a domain or a principal value range; $\cot^{-1}(-1)$
  - a ratio of an inverse: $\sin(\cos^{-1}(3/5))$, $\tan(\sin^{-1}(-5/13))$
  - a sum of inverse tangents; x from one
  - which identity fails outside its condition; $\sin^{-1}(\sin 10)$
- **Short (5 marks, in parts)**
  - domain, range and graph of an inverse ($\sin^{-1}$, cosec^{-1}, $\cos^{-1}$ from its function's graph)
  - principal values
  - prove a sum of inverse sines or tangents is $\pi/2$, $\pi/4$
  - solve an equation in inverse tangents
  - prove a double-angle form and use it; $x + y + z = xyz$

Key terms

Inverse sine $\sin^{-1} x$ (arcsin x): the angle in $[-\pi/2, \pi/2]$ whose sine is x.

Principal value: the one angle the inverse gives; its range is the **principal value branch**.

Circular: an angle in radians is an arc of the unit circle.

$\sin^{-1} x$ is an **angle**, never $1/\sin x$ (that is cosec x); but $\sin^2 x = (\sin x)^2$.

- a circular function is not one-to-one → **restrict the domain** to a piece that takes every value once and holds the acute angles (as $\sqrt{x}$ from x^2, $x \geq 0$)

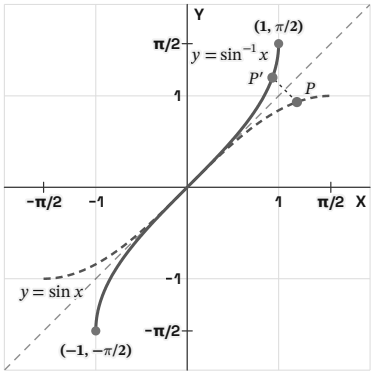
$$y = \sin^{-1} x \Leftrightarrow \sin y = x \text{ and } -\pi/2 \leq y \leq \pi/2$$

Function	Domain	Principal values
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$\mathbb{R}$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$\mathbb{R}$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \{\pi/2\}$
$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, \pi/2] - \{0\}$

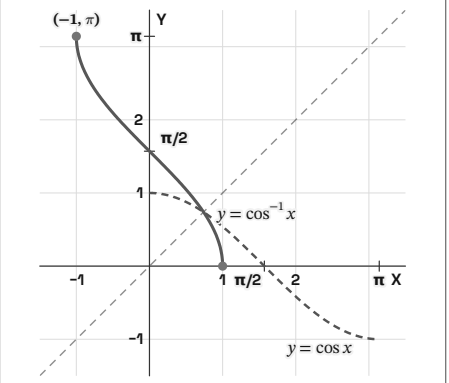
- **sin, tan, cosec**: right half of the circle; **cos, cot, sec**: upper half; each without its undefined angles
- $\cot^{-1}$ in $(0, \pi)$: $\cot^{-1}(-1) = 3\pi/4$ (some books $-\pi/4$)
- $\sin^{-1}(1/2) = \pi/6$; $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$, not $4\pi/3$
- **a ratio of an inverse**: draw the right triangle; $\cos(\sin^{-1}(3/5)) = 4/5$; $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$, never negative

Graphs

Mirror image of the principal branch in $y = x$: domain and range swap.



- $\sin^{-1} x$: $(-1, -\pi/2)$ to $(1, \pi/2)$
- increasing, odd
- vertical tangent at the ends



- $\cos^{-1} x$: $(-1, \pi)$ down to $(1, 0)$, through $(0, \pi/2)$
- decreasing; symmetric about $(0, \pi/2)$

- $\tan^{-1} x$: all x; increasing, odd; horizontal asymptotes $y = \pm\pi/2$
- $\cot^{-1} x$: falls from near π to near 0; asymptotes $y = 0, y = \pi$
- $\sec^{-1} x$: two pieces, from $(1, 0)$ and $(-1, \pi)$; asymptote $y = \pi/2$
- $\operatorname{cosec}^{-1} x$: two pieces; asymptote $y = 0$

Properties

Undoing

- $\sin(\sin^{-1} x) = x$ for all $|x| \le 1$
- $\sin^{-1}(\sin\theta) = \theta$ **only** for θ in the branch; else bring it back: $\sin^{-1}(\sin(5\pi/6)) = \pi/6$

Negatives

- $\sin^{-1}(-x) = -\sin^{-1} x$; same for tan, cosec (odd)
- $\cos^{-1}(-x) = \pi - \cos^{-1} x$; same for cot, sec
- proof: $\theta = \cos^{-1} x$; $\cos(\pi - \theta) = -x$; $\pi - \theta$ in $[0, \pi]$

Reciprocals ($|x| \ge 1$)

- $\operatorname{cosec}^{-1} x = \sin^{-1}(1/x)$; $\sec^{-1} x = \cos^{-1}(1/x)$
- $\cot^{-1} x = \tan^{-1}(1/x)$ for $x > 0$; $\pi + \tan^{-1}(1/x)$ for $x < 0$

Complementary pairs

- $\sin^{-1} x + \cos^{-1} x = \pi/2$ ($|x| \le 1$)
- $\tan^{-1} x + \cot^{-1} x = \pi/2$ (all x)
- $\sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2$ ($|x| \ge 1$)
- proof: $\theta = \sin^{-1} x$; $\cos(\pi/2 - \theta) = x$; $\pi/2 - \theta$ in $[0, \pi]$

One in terms of another: draw the triangle (opposite x, hypotenuse 1)

- $\sin^{-1} x = \cos^{-1}\sqrt{1 - x^2}$ $(0 \le x \le 1)$ $= \tan^{-1}(x/\sqrt{1 - x^2})$ $(|x| < 1)$

Double and triple angles

Put $x = \tan\theta$ (or $\sin\theta$, $\cos\theta$); the condition keeps the multiple angle in its branch.

Identity	Holds for
$2\tan^{-1} x = \sin^{-1}\dfrac{2x}{1+x^2} = \tan^{-1}\dfrac{2x}{1-x^2}$	$-1 \le x \le 1$ (tan form: $-1 < x < 1$)
$2\tan^{-1} x = \cos^{-1}\dfrac{1-x^2}{1+x^2}$	$x \ge 0$
$2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$	$-1/\sqrt{2} \le x \le 1/\sqrt{2}$
$2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$	$0 \le x \le 1$
$3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$	$-\dfrac{1}{2} \le x \le \dfrac{1}{2}$
$3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$	$\dfrac{1}{2} \le x \le 1$
$3\tan^{-1} x = \tan^{-1}\dfrac{3x - x^3}{1 - 3x^2}$	$-1/\sqrt{3} < x < 1/\sqrt{3}$

- proof of the first: $\theta = \tan^{-1} x$, $|\theta| \le \pi/4$; $\sin 2\theta = 2x/(1 + x^2)$; 2θ in the branch
- outside the condition the identity is **false**: $2\tan^{-1} 2 + \sin^{-1}(4/5) = \pi$

Sums and differences

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy} \qquad (xy < 1)$$

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy} \qquad (xy > -1)$$

- proof: $A = \tan^{-1} x$, $B = \tan^{-1} y$; $\tan(A + B)$; $xy < 1$ keeps $A + B$ in the branch
- $xy > 1$: add π ($x, y > 0$) or subtract π ($x, y < 0$); $\tan^{-1} 2 + \tan^{-1} 3 = 3\pi/4$
- $xy = 1, x > 0$: $\tan^{-1} x + \tan^{-1}(1/x) = \pi/2$

Sines and cosines (from $\sin(A \pm B)$, $\cos(A \pm B)$; $\sqrt{1-x^2}$ for the other ratio)

$$\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$$

$$\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right)$$

- sum of sines: $x, y \geq 0, x^2 + y^2 \leq 1$; difference: $0 \leq x, y \leq 1$
- cosines: $0 \leq x, y \leq 1$; difference $x \leq y$
- **equations**: take tan (or sin) of both sides; **check every root** in the original
- three inverse tangents adding to π: $x + y + z = xyz$

Machin: $\pi/4 = 4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239}$

1. $2\tan^{-1}(1/5) = \tan^{-1}(5/12)$
2. $4\tan^{-1}(1/5) = \tan^{-1}(120/119)$
3. difference rule: $\tan^{-1} 1 = \pi/4$

- fast: each term of the series at $1/5$ about $1/25$ of the last; at $1/239$ about $1/57\,000$

COMMON MISTAKES

✗ $\sin^{-1} x = 1/\sin x$
✓ an angle; $1/\sin x$ is cosec x.

✗ $\sin^{-1}(\sin(5\pi/6)) = 5\pi/6$
✓ $\pi/6$: back into the branch.

✗ $\cos^{-1}(-x) = -\cos^{-1} x$
✓ $\pi - \cos^{-1} x$.

✗ $\cos^{-1}$ range $[-\pi/2, \pi/2]$
✓ $[0, \pi]$.

✗ $\tan^{-1} x + \tan^{-1} y$ formula when $xy > 1$
✓ add π.

✗ an identity used outside its condition
✓ the condition is part of it.

✗ every root of the tangent equation kept
✓ check each in the original.