Unit 8 · Complex Number

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Algebra (Units 1–8 together): 20 of 75 marks
  - two 1-mark choices
  - two 5-mark short questions
  - one 8-mark long question
- **MCQ (1 mark)**
  - a power of i (i^{51}); a modulus; a conjugate as a reflection
  - a product or quotient; $\bar{z} = -z$ means purely imaginary
  - x, y from $x + yi = (1 + i)^2$; a distance on the Argand plane
  - the points with $|z - 1| = |z + 1|$
- **Short (5 marks, in parts)**
  - real x, y from one complex equation
  - $(1 + i)/(1 - i) = i$ and its powers
  - square roots of $7 - 24i$; solve a quadratic with them
  - prove a conjugate property; $(x + yi)^{1/3} = a + bi$ results
  - $|z - 3|^2 + |z + 3|^2$ on the Argand plane
- **Long (8 marks)**
  - plot z_1, z_2, $z_1 + z_2$; moduli, triangle inequality, z_1/z_2
  - $|z|$, $\bar{z}$, $1/z$, powers, square roots of one number

Key terms

Imaginary unit i: $i^2 = -1$ (engineers write j).

Complex number $z = a + bi$, a, b real: **real part** $\mathrm{Re}(z) = a$, **imaginary part** $\mathrm{Im}(z) = b$ (not bi).

Real: $b = 0$ (every real number is complex). **Purely imaginary**: $a = 0$, $b \neq 0$.

Argand plane (complex plane): $a + bi$ is the point (a, b); real axis across, imaginary axis up.

Modulus $|z| = \sqrt{a^2 + b^2}$: distance from O.

Conjugate $\bar{z} = a - bi$: reflection in the real axis.

Powers of i

$$i^{4n} = 1, \quad i^{4n+1} = i, \quad i^{4n+2} = -1, \quad i^{4n+3} = -i$$

- cycle $i, -1, -i, 1$: divide the power by 4, use the remainder
- $\sqrt{-a} = i\sqrt{a}$ $(a > 0)$: $\sqrt{-9} = 3i$, $\sqrt{-121} = 11i$
- $\sqrt{-4} \times \sqrt{-9} = 2i \times 3i = -6$, **not** $\sqrt{36}$: change to i first

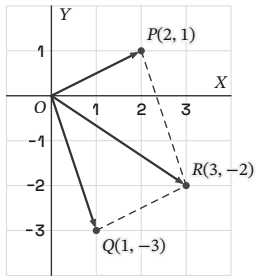
Algebra

- **equal**: $a + bi = c + di \Leftrightarrow a = c$ and $b = d$: one complex equation = two real ones
- add, subtract: part by part
- multiply: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
- divide: top and bottom × the conjugate of the bottom; $(c + di)(c - di) = c^2 + d^2$

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{c^2 + d^2}$$

- all the rules of algebra hold; every $z \neq 0$ has a reciprocal
- **no order**: $z_1 > z_2$ means nothing ($i > 0$ or $i < 0$ would make i^2 positive)
- $(1 + i)/(1 - i) = i$

On the Argand plane



- $z_1 = 2 + i, z_2 = 1 - 3i$
- $z_1 + z_2$: the parallelogram's far corner

- z as the arrow from O to (a, b): added like vectors
- $z - w$: the arrow from w to z
- iz: $(a, b) \to (-b, a)$, a quarter turn anticlockwise; twice: $-z$
- $|z_1 - z_2|$: **distance** between two points
- $|z - (1 + i)| = 2$: circle, center (1, 1), radius 2; $(x - 1)^2 + (y - 1)^2 = 4$

Modulus and conjugate: properties

1. $\bar{\bar{z}} = z$; z real $\Leftrightarrow \bar{z} = z$; purely imaginary $\Leftrightarrow \bar{z} = -z$
2. $z + \bar{z} = 2\,\mathrm{Re}(z)$; $z - \bar{z} = 2i\,\mathrm{Im}(z)$
3. $z\bar{z} = a^2 + b^2 = |z|^2$: real, never negative
4. $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$, $\overline{(z_1/z_2)} = \bar{z}_1/\bar{z}_2$
5. $|\bar{z}| = |-z| = |z|$
6. $|z_1 z_2| = |z_1||z_2|$; $|z_1/z_2| = |z_1|/|z_2|$
7. **triangle inequality** $|z_1 + z_2| \le |z_1| + |z_2|$; equal only when they point the same way, or one is 0

Property 6 from 3

1. $|z_1 z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)}$
2. $= z_1 z_2 \bar{z}_1 \bar{z}_2$ (property 4)
3. $= (z_1\bar{z}_1)(z_2\bar{z}_2) = |z_1|^2|z_2|^2$; take square roots

Square root of a + bi

1. $(x + yi)^2 = a + bi \to x^2 - y^2 = a$, $2xy = b$
2. moduli: $x^2 + y^2 = \sqrt{a^2 + b^2}$
3. add, subtract: $x^2 = (\sqrt{a^2 + b^2} + a)/2$, $y^2 = (\sqrt{a^2 + b^2} - a)/2$
4. signs: $b > 0$ same; $b < 0$ opposite
5. answer $\pm(x + yi)$; check by squaring

- every $z \neq 0$: two square roots, negatives of each other; 0: only 0
- real quadratic with $b^2 - 4ac < 0$: two **conjugate** roots; $x^2 + 2x + 5 = 0 \to -1 \pm 2i$
- **Bombelli**: $x^3 = 15x + 4$; $(2 + i)^3 = 2 + 11i$, $(2 - i)^3 = 2 - 11i \to x = (2 + i) + (2 - i) = 4$: imaginary parts cancel
- Grade 12: polar form, De Moivre's theorem

COMMON MISTAKES

✗ $\sqrt{-4}\sqrt{-9} = \sqrt{36} = 6$
✓ $2i \times 3i = -6$.

✗ $\mathrm{Im}(3 + 2i) = 2i$
✓ 2: a real number.

✗ $a + bi = c + di$ gives one equation
✓ two: real parts and imaginary parts.

✗ $|a + bi| = \sqrt{a^2 - b^2}$
✓ $\sqrt{a^2 + b^2}$.

✗ $3 + 2i > 1 + i$
✓ complex numbers have no order.

✗ one square root only
✓ two: $\pm(x + yi)$.

✗ signs of x, y chosen freely
✓ fixed by $2xy = b$.

Science Playground, Katunje, Bhaktapur
https://science-playground.com/learn/grade-11-math/complex-number