Unit 7 · Quadratic Equation

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Algebra (Units 1–8 together): 20 of 75 marks
  - two 1-mark choices
  - two 5-mark short questions
  - one 8-mark long question
- **MCQ (1 mark)**
  - the discriminant; the sum of the roots; the nature of roots
  - the equation with given roots; reciprocal roots
  - $1/\alpha + 1/\beta$; k for equal roots; the other root from one
  - the condition for real roots
- **Short (5 marks, in parts)**
  - prove roots real or not real (a sum of squares)
  - form an equation with roots made from old ones (α/β, α^2)
  - a common root and the value of k; both roots common
  - one root twice the other: $2b^2 = 9ac$
- **Long (8 marks)**
  - k for equal roots and the roots; roots squared from $x^2 - px + q = 0$

Key terms

Quadratic equation: $ax^2 + bx + c = 0$, a, b, c real, $a \neq 0$; its solutions are **roots**.

Discriminant $D = b^2 - 4ac$: decides the nature of the roots.

Symmetric expression: unchanged when α and β swap ($\alpha^2 + \beta^2$; not $\alpha - \beta$).

Polynomial equation of degree n: $a_n x^n + \cdots + a_1 x + a_0 = 0$, whole-number powers, $a_n \neq 0$.

Formulas

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$D = b^2 - 4ac$ **Use:** the roots
$\alpha + \beta = -b/a$, $\alpha\beta = c/a$	α, β the roots, real or not **Use:** sum and product
$x^2 - (\alpha + \beta)x + \alpha\beta = 0$	sum and product of the roots **Use:** forming an equation
$(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$	coefficients of the two equations **Use:** one common root
$a_1/a_2 = b_1/b_2 = c_1/c_2$	the same $(x - \alpha)(x - \beta)$ **Use:** both roots common

Nature of the roots

The formula, by completing the square

1. divide by a: $x^2 + (b/a)x = -c/a$
2. add $b^2/4a^2$ to both sides
3. $(x + b/2a)^2 = (b^2 - 4ac)/4a^2$

4. $D \geq 0$: square roots → the formula; $D < 0$: a square can't be negative

D	Roots	Graph of $y = ax^2 + bx + c$
> 0	real, unequal	crosses the X-axis twice
$= 0$	real, equal, $-b/2a$	touches at its vertex
< 0	not real (complex, Unit 8)	misses the X-axis

- a, b, c rational, $D > 0$: D a perfect square → rational roots; else irrational pair $p \pm \sqrt{q}$
- **prove roots real**: show $D \geq 0$, often as a sum of squares
- $x^2 - 2(a+b)x + 2(a^2+b^2) = 0$: $D/4 = -(a-b)^2$ → not real unless $a = b$ (then both $2a$)
- **kitchen garden**, 40 m of wire: $x^2 - 20x + A = 0$, $D = 400 - 4A$
  - 96 m^2: $D = 16$ → 12 m by 8 m
  - 120 m^2: $D = -80$ → impossible
  - largest at $D = 0$: 100 m^2, a 10 m square

Roots and coefficients

- from the formula: add → $-2b/2a$; multiply → $(b^2 - D)/4a^2 = c/a$
- or: $a(x-\alpha)(x-\beta) = a[x^2 - (\alpha+\beta)x + \alpha\beta]$, compare coefficients
- $c/a < 0$: real roots of **opposite signs**
- real roots, $c/a > 0$: same sign, the sign of $-b/a$
- a root given: the other from the sum or product

Forming an equation

- $x^2 - (\text{sum})x + (\text{product}) = 0$; multiply to clear fractions
- new roots (squares, reciprocals, $\alpha + 2$): their sum and product from $\alpha + \beta$, $\alpha\beta$; never solve

Symmetric expressions

- $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
- $(\alpha - \beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = D/a^2$
- $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$
- $1/\alpha + 1/\beta = (\alpha+\beta)/\alpha\beta$
- $\alpha/\beta + \beta/\alpha = (\alpha^2+\beta^2)/\alpha\beta$

Special pairs

- equal and opposite ($\beta = -\alpha$): $b = 0$
- reciprocals ($\beta = 1/\alpha$): $c = a$
- one k times the other: $(1+k)\alpha = -b/a$, $k\alpha^2 = c/a$ → $kb^2 = (1+k)^2\, ac$ (twice: $2b^2 = 9ac$)

Common roots

1. α common: $a_1\alpha^2 + b_1\alpha + c_1 = 0$, $a_2\alpha^2 + b_2\alpha + c_2 = 0$
2. α^2 and α as two unknowns: cross-multiplication
3. $\alpha = (c_1a_2 - c_2a_1)/(a_1b_2 - a_2b_1)$; $\alpha^2 = (b_1c_2 - b_2c_1)/(a_1b_2 - a_2b_1)$
4. square of the first = the second: the condition

- quicker: make the x^2 terms equal, **subtract** → a linear equation for the common root; put it back
- $a_1b_2 - a_2b_1 = 0$: a common root needs both roots common

Polynomial equations

- **fundamental theorem of algebra**: every polynomial equation of degree $n \geq 1$ has a complex root; so exactly n roots, counting repeats
- why: a root α_1 → factor $x - \alpha_1$ → degree $n-1$ … → $a_n(x-\alpha_1)\cdots(x-\alpha_n)$
- Gauss 1799 (gap filled 1920); Argand, first rigorous proof 1806; state and use it
- real coefficients: non-real roots in **conjugate pairs**; odd degree → at least one real root; at most n real roots

For the IOE entrance

- remainder theorem: $f(x) \div (x-a)$ leaves $f(a)$; by $ax - b$: $f(b/a)$
- factor theorem: $x - a$ a factor $\Leftrightarrow f(a) = 0$
- cubic $ax^3 + bx^2 + cx + d = 0$: $\Sigma\alpha = -b/a$, $\Sigma\alpha\beta = c/a$, $\alpha\beta\gamma = -d/a$ (signs alternate)
- $\alpha^2 + \beta^2 + \gamma^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$; negative → not all roots real
- roots in AP: $m-d$, m, $m+d$; $2x^3 - 15x^2 + 37x - 30 = 0 \to 2, 5/2, 3$

COMMON MISTAKES

✗ $\alpha + \beta = b/a$
✓ $-b/a$.

✗ equal roots when $D > 0$
✓ $D = 0$; $D > 0$: unequal.

✗ $D < 0$: no roots
✓ no real roots; two complex ones.

✗ solving for α, β to form a new equation
✓ sum and product only.

✗ $(\alpha - \beta)^2 = (\alpha+\beta)^2 - 2\alpha\beta$
✓ $-4\alpha\beta$.

✗ reciprocal roots need $b = 0$
✓ $c = a$; $b = 0$ is equal and opposite.

✗ k found, not checked
✓ k must keep it a quadratic ($a \neq 0$).