# Unit 6 · Matrices and Determinants

STUDY NOTES

**WHAT NEB'S GRADE 11 EXAM ASKS**

- **CDC's grid (2078):** Algebra (Units 1–8 together): 20 of 75 marks
  - two 1-mark choices
  - two 5-mark short questions
  - one 8-mark long question
- **MCQ (1 mark)**
  - a cofactor's formula or value; a skew-symmetric matrix and its determinant
  - a triangular determinant; the effect of a swap and a multiplied row
  - a $2 \times 2$ adjoint; $|\text{adj } A|$; $k$ for no inverse; $A^3 = 2I \to |A|$
- **Short (5 marks, in parts)**
  - symmetric + skew-symmetric; verify $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$; $AA^{\mathrm{T}}$ symmetric
  - a $3 \times 3$ inverse; from $A^2 - 5A + 4I = O$
  - show a determinant is 0 without expanding; solve a determinant equation
  - a triangle's area by a determinant; collinear points
- **Long (8 marks)**
  - adj $A$, $A^{-1}$ with its check, $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$
  - factorize a determinant by its properties

## Key terms

**Transpose** $A^{\mathrm{T}}$ (or $A'$): rows become columns; $(A^{\mathrm{T}})_{ij} = a_{ji}$; $m \times n \to n \times m$.

**Symmetric**: $A^{\mathrm{T}} = A$ ($a_{ji} = a_{ij}$).

**Skew-symmetric**: $A^{\mathrm{T}} = -A$; diagonal all zeros.

**Minor** $M_{ij}$: the determinant left when row $i$ and column $j$ are struck out.

**Cofactor** $C_{ij} = (-1)^{i+j}M_{ij}$.

**Adjoint** adj $A$: the transpose of the matrix of cofactors ($C_{ij}$ goes to place $(j, i)$).

**Singular**: $|A| = 0$, no inverse. **Non-singular**: $|A| \neq 0$.

## Transpose

- $(A^{\mathrm{T}})^{\mathrm{T}} = A$; $(A \pm B)^{\mathrm{T}} = A^{\mathrm{T}} \pm B^{\mathrm{T}}$; $(kA)^{\mathrm{T}} = kA^{\mathrm{T}}$
- **reversal law** $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$
  1. $((AB)^{\mathrm{T}})_{ij} = (AB)_{ji} = \Sigma_k\, a_{jk}b_{ki}$
  2. $(B^{\mathrm{T}}A^{\mathrm{T}})_{ij} = \Sigma_k\, b_{ki}a_{jk}$
  3. same terms $\to$ equal
- order turns round: $A^{\mathrm{T}}B^{\mathrm{T}}$ can't even be multiplied unless $m = n$ (socks and shoes)
- $A + A^{\mathrm{T}}$ symmetric; $A - A^{\mathrm{T}}$ skew-symmetric

$$A = \frac{1}{2}(A + A^{\mathrm{T}}) + \frac{1}{2}(A - A^{\mathrm{T}})$$

- every square matrix: symmetric + skew-symmetric, in one way only ($A = P + Q \to A^{\mathrm{T}} = P - Q$)

## Determinant

**Checkerboard of signs** (+ top left):

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

**Expanding along the first row**

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{aligned} |A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

- any row or column gives the same: pick the one with most zeros
- $2 \times 2$: $ad - bc$
- **rule of Sarrus** (copy two columns, diagonals): $3 \times 3$ only; cofactors work for every size
- meaning: $|\det A|$ = volume of the box (parallelepiped) the unit cube goes to; negative: turned inside out; 0: flat, no inverse

## Properties (stated without proof)

Row can be read as column in every one.

1. $|A^{\mathrm{T}}| = |A|$
2. **swap** two rows: sign changes
3. two **equal** or proportional rows, or a row of zeros: 0
4. a row $\times\, k$: determinant $\times\, k$; a common factor comes out; $|kA| = k^3|A|$ $(3 \times 3)$
5. a row that is a **sum** splits into two determinants, the other rows unchanged
6. $R_i \to R_i + kR_j$: **unchanged** (the workhorse: make zeros, expand)
7. **triangular**: product of the diagonal
8. $|AB| = |A||B|$

- skew-symmetric $3 \times 3$: $|A| = 0$
- $|A + B| \neq |A| + |B|$: property 5 splits one row at a time

## Adjoint and inverse

$2 \times 2$: swap the diagonal, negate the rest

$$\operatorname{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A(\operatorname{adj} A) = (\operatorname{adj} A)A = |A|I$$

1. diagonal: row $i \times$ its own cofactors $= |A|$ expanded along row $i$
2. off the diagonal: row $i \times$ row $j$'s cofactors $=$ a determinant with two equal rows $= 0$
3. so $|A|$ down the diagonal, 0 elsewhere

$$A^{-1} = \frac{1}{|A|}\operatorname{adj} A, \qquad |A| \neq 0$$

- singular: $AB = I$ would give $|A||B| = 1$, impossible
- $|\operatorname{adj} A| = |A|^2$ $(3 \times 3)$
- $(A^{-1})^{-1} = A$; $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$; $|A^{-1}| = 1/|A|$
- $(AB)^{-1} = B^{-1}A^{-1}$: $(AB)(B^{-1}A^{-1}) = AIA^{-1} = I$
- **check** an inverse: $AA^{-1} = I$
- from $A^2 - 5A + 4I = O$: multiply by $A^{-1} \to A - 5I + 4A^{-1} = O$

**Inverse, steps**

1. $|A|$ (stop if 0)
2. all nine cofactors
3. transpose $\to$ adj $A$
4. divide by $|A|$; check

## Area of a triangle

$$\Delta = \frac{1}{2}\left|\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right|$$

- why: $R_2 - R_1$, $R_3 - R_1 \to$ the $2 \times 2$ determinant of two sides = parallelogram = twice the triangle
- moving the origin never changes the area
- determinant 0: the three points are **collinear**
- plot $A(2, 1)$, $B(26, 3)$, $C(28, 21)$, $D(6, 19)$: cut along $AC$: $214 + 194 = 408$ m$^2$ $\approx 12.8$ aana (1 aana $\approx 31.80$ m$^2$)

Grade 12: $AX = B$ solved by $X = A^{-1}B$ or Cramer's rule.

**COMMON MISTAKES**

✗ $(AB)^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$
✓ $B^{\mathrm{T}}A^{\mathrm{T}}$: the order reverses.

✗ adj $A$ = the matrix of cofactors
✓ its **transpose**.

✗ cofactor = minor
✓ $(-1)^{i+j} \times$ minor: checkerboard signs.

✗ $|2A| = 2|A|$ $(3 \times 3)$
✓ $2^3|A| = 8|A|$.

✗ $A^{-1}$ of a singular matrix
✓ none: $|A| = 0$.

✗ Sarrus for a $4 \times 4$
✓ cofactor expansion.

✗ a swap leaves the determinant
✓ changes its sign.