Unit 5 · Sequence and Series

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Algebra (Units 1–8 together): 20 of 75 marks
  - two 1-mark choices
  - two 5-mark short questions
  - one 8-mark long question
- **MCQ (1 mark)**
  - a term of an AP or GP; the H.M. formula
  - a, b, c in GP means…; in AP and GP at once
  - a sum to infinity; how many terms make a sum
  - $A > G > H$; two numbers from their A.M. and G.M.
- **Short (5 marks, in parts)**
  - an AP from S_n $(3n^2 + 2n)$; a block of its terms
  - insert harmonic means
  - sum of $0.7 + 0.77 + 0.777 + \cdots$
  - prove a property of an HP
  - prove $A \geq G \geq H$; numbers from G and H
  - a sum to infinity; a recurring decimal as a fraction
- **Long (8 marks)**
  - a GP in a real setting (doses of a medicine): S_n, the level it settles at
  - three numbers in GP from their sum and sum of squares; sum from the $(p+1)$th term

Key terms

Sequence: numbers in a definite order; a function on the natural numbers. **Terms** $t_1, t_2, \ldots$; **general term** t_n.

Finite / infinite: stops / doesn't.

Series: the sum of the terms; $S_n = t_1 + \cdots + t_n = \Sigma_{k=1}{}^{n}\, t_k$.

AP (arithmetic): each term the one before **plus** d, the **common difference**.

GP (geometric): each term the one before **times** r, the **common ratio** ($a \neq 0$, $r \neq 0$).

HP (harmonic): the **reciprocals** are in AP.

Converges: S_n has a limit, the sum of the infinite series.

Formulas

$t_n = a + (n-1)d$	a first term, d common difference **Use:** AP's nth term
$S_n = \frac{n}{2}(a + l) = \frac{n}{2}[2a + (n-1)d]$	l last term **Use:** AP's sum
$t_n = ar^{n-1}$	r common ratio **Use:** GP's nth term
$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$	$r \neq 1$; $r = 1$: $S_n = na$ **Use:** GP's sum (second form for $r < 1$)
$S_\infty = \frac{a}{1 - r}$	only when $\lvert r \rvert < 1$ **Use:** infinite GP
$t_n = S_n - S_{n-1}$	S_n given **Use:** a term from the sums
$A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$	$a, b > 0$ **Use:** the three means

Arithmetic progression

Sum, proof

1. $S_n = a + (a+d) + \cdots + l$
2. backwards: $S_n = l + (l-d) + \cdots + a$
3. add: n pairs, each $a + l \to 2S_n = n(a+l)$

- number of terms × average of first and last
- add or multiply every term by a number: still an AP
- $t_k + t_{n+1-k} = a + l$
- a, b, c in AP $\Leftrightarrow 2b = a + c$
- three terms $a - d, a, a + d$; four $a - 3d, a - d, a + d, a + 3d$

Geometric progression

Sum, proof

1. $S_n = a + ar + \cdots + ar^{n-1}$
2. $rS_n = ar + \cdots + ar^{n-1} + ar^n$
3. subtract: $rS_n - S_n = ar^n - a$

- multiply terms by a number, raise to a power, take reciprocals (ratio $1/r$): still a GP
- $t_k \cdot t_{n+1-k} = a \cdot l$
- a, b, c in GP $\Leftrightarrow b^2 = ac$
- three terms $a/r, a, ar$ (product a^3)
- sum of n terms from the $(p+1)$th $= r^p \times$ sum of the first n

EMI (loan P, monthly rate i, n months): the installments valued today form a GP, first term $E/(1+i)$, ratio $1/(1+i)$

$$E = \frac{Pi}{1-(1+i)^{-n}} = \frac{Pi(1+i)^n}{(1+i)^n - 1}$$

- Rs 20,00,000, 10% a year, 15 years: Rs 21,492 a month

Harmonic progression

- no sum formula: work with the AP of the reciprocals
- nth term: $1/[1/a + (n-1)d]$, d the reciprocals' difference
- a, b, c in HP $\Leftrightarrow 2/b = 1/a + 1/c \Leftrightarrow b = 2ac/(a+c)$

A.M., G.M., H.M.

- a, A, b in AP: A the A.M.; in GP: G; in HP: H
- equal distances at two speeds average **harmonically**: 40 and 60 km/h → 48 km/h
- growth rates average **geometrically**: +10% then +30% ≈ 19.6% a year, not 20%

Inserting n means between a and b

- AP: $b = a + (n+1)d \to d = (b-a)/(n+1)$
- GP: $b = ar^{n+1} \to r = (b/a)^{1/(n+1)}$
- HP: arithmetic means between $1/a$ and $1/b$, then reciprocals

$A \geq G \geq H$ $(a, b > 0)$

1. $A - G = (a - 2\sqrt{ab} + b)/2 = (\sqrt{a} - \sqrt{b})^2/2 > 0$ for $a \neq b$
2. $AH = (a+b)/2 \cdot 2ab/(a+b) = ab = G^2$
3. $G/H = A/G > 1 \to G > H$

- equal only when $a = b$; G is the G.M. of A and H
- **positive numbers only**: $-1, -4$: G.M. -2, and $A < G < H$; opposite signs: no real G.M.

Infinite geometric series

1. $S_n = a/(1-r) - r^n \cdot a/(1-r)$
2. $|r| < 1$: $r^n \to 0$
3. so $S_n \to a/(1-r)$

- $|r| \geq 1$: no sum (grows, or swings: $1 - 1 + 1 - \cdots$)
- $1/2 + 1/4 + 1/8 + \cdots = 1$; $1/4 + 1/16 + \cdots = 1/3$
- **recurring decimal**: $0.\overline{7} = (7/10)/(1 - 1/10) = 7/9$
- medicine, 500 mg every 8 h, a quarter left: after n doses $(2000/3)(1 - 1/4^n)$ mg → settles at 2000/3 mg

COMMON MISTAKES

✗ S_∞ for any GP
✓ only when $|r| < 1$.

✗ HP summed like an AP
✓ no sum formula; use the reciprocals.

✗ $H = (a+b)/2ab$
✓ $H = 2ab/(a+b)$.

✗ $G = \sqrt{ab}$ for negatives
✓ positive numbers only.

✗ n means: $d = (b-a)/n$
✓ $(b-a)/(n+1)$.

✗ 40 and 60 km/h average 50
✓ 48: harmonic mean.

✗ $t_n = S_n$
✓ $t_n = S_n - S_{n-1}$.