Unit 4 · Curve Sketching

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Algebra (Units 1–8 together): 20 of 75 marks
  - two 1-mark choices
  - two 5-mark short questions
  - one 8-mark long question
- **MCQ (1 mark)**
  - the odd or even test; which function is even
  - a period ($\tan x$, $5\cos(2x/5)$, $\sin^4 x + \cos^4 x$)
  - a curve's symmetry; e^{-x} as a reflection
  - where a parabola decreases; its vertex; greatest value of $4\sin 3x - 1$
  - how many solutions, from two sketches
- **Short (5 marks, in parts)**
  - even, odd or neither (log, exponential, polynomial)
  - periods, including a sum
  - symmetry about the axes and the origin
  - sketch a parabola, $1/(ax+b)$, $a\sin bx$, e^x and $\ln x$ with intercepts and asymptotes
  - increasing from the definition
- **Long (8 marks)**
  - a cubic: odd/even, intercepts, signs, ends, sketch, roots read off
  - two cosine waves on one sketch and where they meet

Odd and even functions

Even: $f(-x) = f(x)$; symmetric about the Y-axis.

Odd: $f(-x) = -f(x)$; symmetric about the origin (half a turn).

Neither: most functions; $x^2 + x$: $f(1) = 2, f(-1) = 0$.

Even	Odd	Neither
x^2, x^4, $\lvert x\rvert$	x, x^3, $1/x$	$x+1$, e^x
cos, sec	sin, tan	$\ln x$, $x^2 + x$

- $\ln x$: its domain $x > 0$ has no $-x$, so neither test applies
- x^n: even n even, odd n odd; only even powers (constant $= x^0$) → even polynomial; only odd → odd
- even × even, odd × odd: even; even × odd: odd ($x\sin x$ even, $x^2\sin x$ odd)

Test

1. write $f(-x)$, simplify
2. compare with $f(x)$ and with $-f(x)$
3. neither: one point where $f(-x) \neq f(x)$ and $\neq -f(x)$

Periodicity

Periodic: $f(x+T) = f(x)$ for every x, some $T > 0$.

Period: the **least** such T ($2T$, $3T$ work too).

- sin, cos, sec, csc: 2π; tan, cot: π ($\tan(x+\pi) = (-\sin x)/(-\cos x)$)
- $f(bx)$: period $T/\lvert b\rvert$ (squeezed $\lvert b\rvert$ times); $\sin 3x$: $2\pi/3$; $\cos(x/2)$: 4π; $\tan 2x$: $\pi/2$
- $5\sin 3x$, $2 + \sin 3x$: height changes, period doesn't
- $4\cos(2x/3)$: 3π
- $\lvert\sin x\rvert$: π (negative arches flipped up)
- $\sin^2 x = (1 - \cos 2x)/2$: π
- **sum**: LCM of the periods is a period; show nothing smaller works
  - $\sin 2x + \cos 3x$: LCM of π and $2\pi/3 = 2\pi$
  - one term odd, one even: $f(x) + f(-x)$, $f(x) - f(-x)$ give each term → T a period of each
  - $\sin(x/2) + \cos(x/3)$: 4π, 6π → 12π
- constant: every T works, no least one; $\sin(x^2)$: not periodic

Symmetry of a curve

Unchanged when	Symmetric about	Example
$y \to -y$	X-axis	$y^2 = 4x$
$x \to -x$	Y-axis	$x^2 = 4y$ (even function)
both together	origin	$y = x^3$, $xy = 4$ (odd function)
x and y swapped	line $y = x$	$xy = 4$

- why: the reflected point satisfies the same equation
- about both axes $\to$ about the origin too (the circle)

Monotonicity

Increasing on I: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

Decreasing on I: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Monotonic: only up, or only down. Some books say "strictly"; with $\leq$ flat stretches are allowed.

- $ax + b$: up if $a > 0$, down if $a < 0$; x^3, e^x, $\ln x$: up everywhere; x^2: down on $(-\infty, 0]$, up on $[0, \infty)$
- proof: the sign of $f(x_2) - f(x_1)$ for $x_1 < x_2$ (Unit 18: with f')
- $1/x$ decreases on $(-\infty, 0)$ and on $(0, \infty)$, **not** on its whole domain: give the intervals separately, never joined by $\cup$

Sketching: the steps

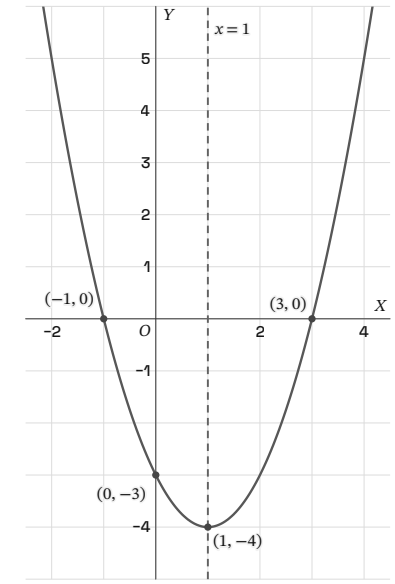
1. **domain**: which x are left out
2. **intercepts**: y at $x = 0$; x where $y = 0$
3. **symmetry, period**: sketch a part, copy it
4. **sign**: above or below the X-axis
5. **asymptotes and ends**: $x \to \pm\infty$
6. **a few values**, joined by a known shape

The known shapes

Parabola $y = ax^2 + bx + c$

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

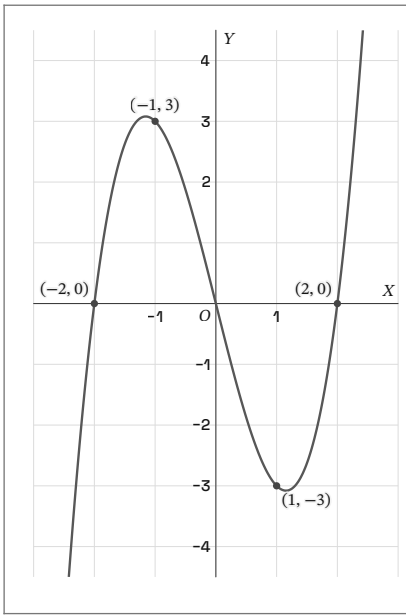
- vertex $x = -b/2a$, $y = -(b^2 - 4ac)/4a$; axis $x = -b/2a$
- $a > 0$ opens up (vertex lowest); $a < 0$ down
- $b^2 - 4ac > 0, = 0, < 0$: meets the X-axis twice, once, never



$y = x^2 - 2x - 3$: crosses at -1 and 3, axis $x = 1$

Cubic $y = ax^3 + bx^2 + cx + d$

- ax^3 wins far out: ends go **opposite ways** ($a > 0$: down left, up right)
- unbroken $\to$ at least one real root
- rises all the way (x^3) or a hump and a dip ($x^3 - 4x$)
- factors give the crossings; signs between them
- repeated factor touches: $x^2(x - 3)$ touches at 0, crosses at 3

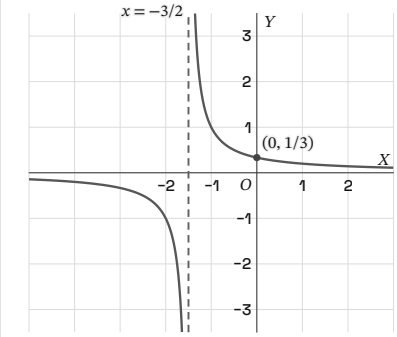


$y = x^3 - 4x$: crosses at -2, 0, 2

Hyperbola $y = 1/(ax + b)$

- $1/x$: odd, two branches; asymptotes $x = 0$, $y = 0$ (**rectangular hyperbola**)
- **asymptote**: a line the curve approaches, never reaches
- vertical asymptote $x = -b/a$; horizontal $y = 0$
- never meets the X-axis; Y-intercept $(0, 1/b)$, $b \neq 0$

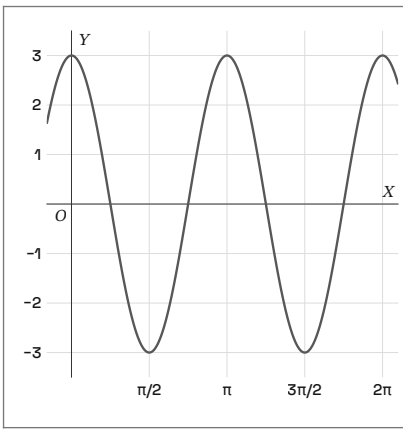
- $a > 0$: decreasing on each branch; $a < 0$: increasing



$y = 1/(2x + 3)$: asymptotes $x = -3/2$, $y = 0$

Waves $y = a\sin bx$, $a\cos bx$

- at $0, \pi/2, \pi, 3\pi/2, 2\pi$: sin 0, 1, 0, −1, 0; cos 1, 0, −1, 0, 1
- **amplitude** $|a|$; range $[-a, a]$; period $2\pi/|b|$
- mark $0, T/4, T/2, 3T/4, T$: sin 0, a, 0, $-a$, 0; cos starts at its top a
- $a < 0$: upside down



$y = 3\cos 2x$: amplitude 3, period π

e^x and $\ln x$

- e^x: all x; range $(0, \infty)$; through $(0, 1)$ and $(1, e)$; increasing; X-axis an asymptote $(x \to -\infty)$
- $\ln x$: $x > 0$; range $\mathbb{R}$; through $(1, 0)$ and $(e, 1)$; increasing; Y-axis an asymptote $(x \to 0^+)$
- mirror images in $y = x$: $y = \ln x$ means $x = e^y$
- e^{-x}: e^x reflected in the Y-axis

Day length in Kathmandu: 10 h 23 min to 13 h 54 min, a sine wave: middle ≈ 12.14 h, amplitude ≈ 1.76 h, period 365 days

$$L \approx 12.14 + 1.76\sin\left(\frac{2\pi(t - 80)}{365}\right)$$

COMMON MISTAKES

✗ $\ln x$ is odd or even
✓ neither: its domain has no $-x$.

✗ period of $\sin 3x$ is 6π
✓ $2\pi/3$: divide by b.

✗ period of a sum: the larger one
✓ the LCM, and show nothing smaller works.

✗ $1/x$ decreasing on $(-\infty, 0) \cup (0, \infty)$
✓ each interval separately.

✗ a sketch as a plot of many points
✓ intercepts, asymptotes, ends, a known shape.

✗ $1/(ax + b)$ meets the X-axis
✓ never: top 1 is never 0.

✗ amplitude of $-2\sin(x/2)$ is -2
✓ 2; the wave is turned over.