Unit 3 · Function

STUDY NOTES

WHAT NEB'S GRADE 11 EXAM ASKS

- **CDC's grid (2078):** Algebra (Units 1–8 together): 20 of 75 marks
  - two 1-mark choices
  - two 5-mark short questions
  - one 8-mark long question
- **MCQ (1 mark)**
  - a domain or a range (e^x, $x^2/(x^2+1)$)
  - $f \circ g$ from two formulas
  - the inverse of 2^x, of a fraction $(ax+b)/(cx+d)$
  - which function has an inverse; where a log graph cuts the x-axis
- **Short (5 marks, in parts)**
  - domain and range (fraction, root, cosine)
  - $f \circ g$ and $g \circ f$; where they are equal
  - f^{-1}; $f \circ f = x$ means $f^{-1} = f$
  - domain of $\ln(x^2 - 5x + 6)$; inverse of a log with its domain and range
- **Long (8 marks)**
  - domain, range, f^{-1}, g^{-1}, $(g \circ f)^{-1}$ two ways

Key terms

Function f: $A \to B$: **each** element of A has **exactly one** image in B.

Domain A, **codomain** B; **range** $f(A)$: the images that really occur.

One-to-one: different inputs, different images. **Onto**: range = codomain.

Vertical line test: a curve is a function of x when no vertical line meets it twice.

Real function: real inputs and outputs, given by a formula.

Natural domain: the largest set of reals for which the formula gives a real number.

Inverse f^{-1}: $f^{-1}(y) = x \Leftrightarrow f(x) = y$; only for one-to-one and onto f.

Composite $g \circ f$: $(g \circ f)(x) = g(f(x))$, f first.

Algebraic: built from x and numbers with $+, -, \times, \div$ and roots.

Transcendental: not algebraic (trigonometric, exponential, logarithmic).

Periodic: values repeat (sin, cos every 2π; tan, cot every π).

Domain and range

Natural domain: leave out every x that gives

- a **denominator** 0
- an **even root** of a negative number
- a **log** of 0 or a negative number
- $(x+1)/(x^2-5x+6)$: $\mathbb{R} - \{2, 3\}$
- $\sqrt{4-x^2}$: $x^2 \le 4 \to [-2, 2]$
- $1/\sqrt{x-3}$: root and division: $x > 3 \to (3, \infty)$

Range

1. write $y = f(x)$
2. solve for x
3. range: every y giving a real x in the domain
4. or complete the square

- $(2x+1)/(x-3)$: $x = (3y+1)/(y-2) \to \mathbb{R} - \{2\}$ (2 approached, never reached)
- $x^2 - 4x + 7 = (x-2)^2 + 3 \to [3, \infty)$
- $1/(1+x^2) \to (0, 1]$
- answers as intervals or $\mathbb{R} - \{\ldots\}$

Inverse function

- domain of f^{-1} = range of f; range of f^{-1} = domain of f
- $f^{-1}(f(x)) = x$, $f(f^{-1}(y)) = y$
- graph of f^{-1}: mirror image of f in $y = x$; $(a, b) \to (b, a)$

To find f^{-1}

1. $y = f(x)$
2. solve for x
3. swap the letters

- **restrict the domain**: x^2 on $\mathbb{R}$ is not one-to-one; on $[0, \infty)$ its inverse is $\sqrt{x}$ (and so the inverse trig functions, Unit 9)
- $f^{-1}(x)$ is **not** $1/f(x)$: $f(x) = 2x$: $f^{-1}(x) = x/2$, $1/f(x) = 1/(2x)$
- $f \circ f = x \to f$ is its own inverse

Composite function

- domain of $g \circ f$: the x in the domain of f with $f(x)$ in the domain of g

- $g \circ f$ and $f \circ g$ usually differ, in formula and domain

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

- undo in reverse order: socks on before shoes, off after
- proof: $(f^{-1} \circ g^{-1})(g(f(x))) = f^{-1}(f(x)) = x$, and the same from the other side

Algebraic functions

Function	Domain	Range
linear $mx + c$, $m \neq 0$	$\mathbb{R}$	$\mathbb{R}$
quadratic $ax^2 + bx + c$	$\mathbb{R}$	from the vertex value $f(-b/2a)$
cubic $ax^3 + bx^2 + cx + d$	$\mathbb{R}$	$\mathbb{R}$

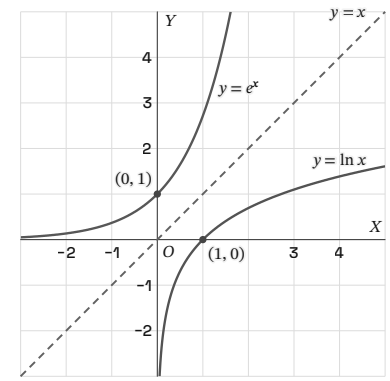
- quadratic: $a > 0$: $[f(-b/2a), \infty)$; $a < 0$: $(-\infty, f(-b/2a)]$
- linear: always one-to-one, always an inverse; quadratic never (the parabola turns)
- cubic: one-to-one only if it never turns back (x^3 yes, $x^3 - 3x$ no)
- $-2x^2 + 8x - 5$: top at $x = 2$, $f(2) = 3$: $(-\infty, 3]$; on $[0, 3]$: ends -5 and $1 \to [-5, 3]$
- a range on a smaller domain: its ends and turning points
- rational functions, roots: algebraic too

Transcendental functions

Function	Domain	Range
sin, cos	$\mathbb{R}$	$[-1, 1]$
tan, sec	$\mathbb{R} - \{\text{odd multiples of } \pi/2\}$	tan $\mathbb{R}$; sec $(-\infty, -1] \cup [1, \infty)$
cot, csc	$\mathbb{R} - \{n\pi\}$	cot $\mathbb{R}$; csc $(-\infty, -1] \cup [1, \infty)$
a^x	$\mathbb{R}$	$(0, \infty)$
$\log_a x$	$(0, \infty)$	$\mathbb{R}$

- x in **radians**: π radians $= 180°$
- a^x: $a > 0$, $a \neq 1$; through $(0, 1)$; $a > 1$ rises, $0 < a < 1$ falls
- equal steps in x multiply by equal factors: $2^{x+1} = 2 \cdot 2^x$ (compound interest, populations, bacteria)
- $e = 2.71828\ldots$; e^x is its own slope (Unit 18)

$$y = \log_a x \Leftrightarrow a^y = x \qquad (x > 0)$$



- e^x and $\ln x$: mirror images in $y = x$
- through $(0, 1)$ and $(1, 0)$

- $\log_{10}$: common; $\log_e x = \ln x$: natural
- laws: $\log_a(xy) = \log_a x + \log_a y$; $\log_a(x/y) = \log_a x - \log_a y$; $\log_a x^k = k \log_a x$
- $\ln(x - 2)$: $(2, \infty)$, range $\mathbb{R}$; $3 - 2^x$: $\mathbb{R}$, range $(-\infty, 3)$; $\log(x^2 - 9)$: $(-\infty, -3) \cup (3, \infty)$
- $5e^{2x} - 1$: $f^{-1}(x) = \frac{1}{2} \ln((x + 1)/5)$, $x > -1$
- **earthquakes**: $\log_{10} E = 1.5M + 4.8$; one magnitude step $\times\ 10^{1.5} \approx 32$; 7.8 vs 7.3: $10^{0.75} \approx 5.6$ times the energy

For the engineering entrance (IOE)

- **injective** (one-one): $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$; no horizontal line meets the graph twice
- **surjective** (onto): every $y \in B$ has an x; range = codomain
- **bijective**: both; exactly these have inverses
- depends on domain and codomain: x^2 on $[0, \infty) \to [0, \infty)$ is bijective; $\mathbb{R} \to \mathbb{R}$ neither
- $5x - 2$ bijective: $x_1 = x_2$ from equal images; $x = (y + 2)/5$ for any y
- counting, A 3 elements, B 4: functions $4^3 = 64$; injective $4 \times 3 \times 2 = 24$; injective $B \to A$: none (pigeonhole)

COMMON MISTAKES

✗ $f^{-1}(x) = 1/f(x)$
✓ the inverse function; $\sin^{-1} x \neq 1/\sin x$ too.

✗ $g \circ f$ means g first
✓ f first: $g(f(x))$.

✗ $(g \circ f)^{-1} = g^{-1} \circ f^{-1}$
✓ $f^{-1} \circ g^{-1}$: reverse order.

✗ domain of $1/\sqrt{x - 3}$ is $x \geq 3$
✓ $x > 3$: no division by 0.

✗ range of $(2x + 1)/(x - 3)$ is $\mathbb{R}$
✓ $\mathbb{R} - \{2\}$: solve for x.

✗ x^2 on $\mathbb{R}$ has inverse $\sqrt{x}$
✓ only on $[0, \infty)$.

✗ log of 0 allowed
✓ input must be > 0.