Unit 6 · Matrix and Determinant

HARDER WORKSHEET · ANSWER KEY

Full marks 25

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Multiple choice

i. **b.** 2×2 U · 1

A^{T} is 2×3, so $A^{\mathrm{T}}A$ is $(2 \times 3)(3 \times 2)$: the inside numbers match and the product is 2×2.

ii. **c.** 16 A · 1

Every element is doubled, so each of the products ad and bc is multiplied by 4: $|2A| = 4|A| = 16$.

iii. **a.** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ A · 1

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. (It swaps the two numbers of a column; swapping twice changes nothing: the reflection in $y = x$.)

2 Short answers

a. **ANSWER** HA · 2

$A + A^{\mathrm{T}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}$. The two elements off the diagonal are equal, so it equals its transpose: symmetric.

MARKING SCHEME

✓ $A + A^{\mathrm{T}} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}$.

✓ Equal elements across the diagonal, so it equals its own transpose.

b. **MARKING SCHEME** U · 2

✓ $(A + B)^2 = A^2 + AB + BA + B^2$.

✓ AB and BA are not equal in general, so they can't be added as $2AB$.

c. **MARKING SCHEME** A · 2

✓ The inverse $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$, multiplied on the left.

✓ $X = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

d. **ANSWER** HA · 2

$kA = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$, so $|kA| = ka \cdot kd - kb \cdot kc = k^2(ad - bc) = k^2|A|$.

MARKING SCHEME

✓ $|kA| = (ka)(kd) - (kb)(kc)$.

✓ $= k^2(ad - bc) = k^2|A|$.

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3 Long answers

a. **ANSWER** **A · 4**

In matrix form $\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 22 \end{pmatrix}$. The determinant is $12 - 10 = 2$, and the inverse $\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}$.

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 48 - 44 \\ -60 + 66 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$: $x = 2, y = 3$.

Check: $6 + 6 = 12$ ✓ and $10 + 12 = 22$ ✓.

MARKING SCHEME

- ✓ The equations in matrix form.
- ✓ The determinant 2 and the inverse.
- ✓ $x = 2, y = 3$.
- ✓ The check in both equations.

b. **ANSWER** **HA · 4**

a. $A^2 = \begin{pmatrix} 9 + 2 & 3 + 4 \\ 6 + 8 & 2 + 16 \end{pmatrix} = \begin{pmatrix} 11 & 7 \\ 14 & 18 \end{pmatrix}$. Then $A^2 + kA + mI = \begin{pmatrix} 11 + 3k + m & 7 + k \\ 14 + 2k & 18 + 4k + m \end{pmatrix} = O$:
$k = -7$, then $11 - 21 + m = 0$, $m = 10$ (and $18 - 28 + 10 = 0$ ✓). So $A^2 - 7A + 10I = O$.

b. $A^2 - 7A = -10I$, so $A(A - 7I) = -10I$ and $A \cdot \frac{1}{10}(7I - A) = I$:

$A^{-1} = \frac{1}{10}(7I - A) = \frac{1}{10}\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$. The formula: $|A| = 12 - 2 = 10$, $A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$, the same.

MARKING SCHEME

- ✓ $A^2 = \begin{pmatrix} 11 & 7 \\ 14 & 18 \end{pmatrix}$.
- ✓ $k = -7$, $m = 10$.
- ✓ $A^{-1} = \frac{1}{10}(7I - A)$ from the equation.
- ✓ $A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$, agreeing with the formula.

c. **ANSWER** **A · 4**

Their product is I: $\begin{pmatrix} 3p + (2p - 9)q & 5p + (2p - 9)p \\ 0 & 3p - 5q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Top right: $2p^2 - 4p = 0$, so $p = 0$ or $p = 2$. Bottom right: $3p - 5q = 1$.

$p = 2$ gives $q = 1$, and the top left is $6 + (4 - 9)(1) = 1$ ✓. $p = 0$ gives $q = -1/5$, but then the top left is $9/5$, not 1: rejected. **$p = 2$, $q = 1$.**

MARKING SCHEME

- ✓ The product set equal to I.
- ✓ The top right element: $2p^2 - 4p = 0$, so $p = 0$ or 2.
- ✓ $3p - 5q = 1$ gives $q = 1$ when $p = 2$.
- ✓ $p = 0$ rejected (the top left element is then $9/5$, not 1): $p = 2$, $q = 1$.

4 Think

ANSWER **HA · 2**

If A were non-singular, it would have an inverse, and then $B = IB = A^{-1}AB = A^{-1}O = O$, which is not so. So A has no inverse: $|A| = 0$, singular. (In Exercise 6.2, $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$ has determinant 0.)

MARKING SCHEME

- ✓ If A^{-1} existed, $B = A^{-1}AB = A^{-1}O = O$.
- ✓ That contradicts $B \neq O$, so A is singular.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	3	12	10	**25**

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