Unit 2 · Polynomial

ANSWER KEY · FOR TEACHERS

Full marks 33

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

1. degree
2. divisor
3. quotient
4. remainder
5. factor
6. synthetic division

Each carries 0.5 mark. K · 3

2 Multiple choice

i. **a.** 4 K · 1

Put $x = 2$: $8 - 8 + 4 = 4$. -12 is $f(-2)$, the remainder on dividing by $x + 2$.

ii. **d.** $x - 1$ U · 1

$f(1) = 1 - 13 + 12 = 0$, so $x - 1$ is a factor. The others leave remainders: $f(-1) = 24$, $f(-2) = 30$, $f(2) = -6$.

iii. **b.** 4 U · 1

Dividing by a divisor of degree 1 lowers the degree by 1. The remainder is a number, of degree 0.

iv. **d.** $\frac{2}{3}$ A · 1

A rational root $\frac{p}{q}$ has p dividing 3 and q dividing 2: the candidates are ± 1, ± 3, $\pm ½$, $\pm ⅔$. $\frac{2}{3}$ has them the wrong way round.

v. **a.** $f\left(\frac{1}{2}\right)$ U · 1

$2x - 1 = 0$ when $x = \frac{1}{2}$: put in the number that makes the divisor zero.

3 Very short answers

a. MARKING SCHEME K · 1

✓ Degree 4 and leading coefficient -3.

b. MARKING SCHEME U · 1

✓ Remainder 4.

c. MARKING SCHEME A · 1

✓ $k = -5$.

4 Short answers

a. MARKING SCHEME U · 2

✓ $x = -2$ should be put in, since $x + 2 = 0$ when $x = -2$.

✓ Remainder -23.

b. MARKING SCHEME A · 2

✓ Quotient $x^2 - x + 2$.

✓ Remainder 0.

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c. MARKING SCHEME U · 2

- ✓ $f(1)$ is the sum of the coefficients, so here $f(1) = 0$.
- ✓ By the factor theorem, $x - 1$ is a factor.

d. MARKING SCHEME A · 2

- ✓ $f(-2) = 0$, so $x + 2$ is a factor.
- ✓ The other factor $x^2 - 2x + 4$.

5 Long answers

a. ANSWER A · 4

The rational root theorem's candidates are $\frac{p}{q}$ with p dividing 3 and q dividing 2: ± 1, ± 3, $\pm ½$, $\pm ⅔$.

$f(1) = 2 - 5 - 4 + 3 = -4$, not 0. $f(-1) = -2 - 5 + 4 + 3 = 0$, so $x + 1$ is a factor.

Synthetic division by -1 of 2, -5, -4, 3 gives 2, -7, 3, 0: the quotient is $2x^2 - 7x + 3 = (2x - 1)(x - 3)$.

So $2x^3 - 5x^2 - 4x + 3 = (x + 1)(2x - 1)(x - 3)$, with roots -1, ½ and 3.

MARKING SCHEME

- ✓ The candidates listed, or a root found by trying them.
- ✓ $f(-1) = 0$, so $x + 1$ is a factor.
- ✓ Quotient $2x^2 - 7x + 3$.
- ✓ $(x + 1)(2x - 1)(x - 3)$.

b. ANSWER A · 4

i. $f(1) = 1 + a + b - 6 = 0$, so $a + b = 5$. $f(-1) = -1 + a - b - 6 = -8$, so $a - b = -1$. Adding: $2a = 4$, $a = 2$, and $b = 3$.

ii. The polynomial is $x^3 + 2x^2 + 3x - 6$. Synthetic division by 1 of 1, 2, 3, -6 gives 1, 3, 6, 0: the quotient is $x^2 + 3x + 6$.

MARKING SCHEME

- ✓ $a + b = 5$ from $f(1) = 0$.
- ✓ $a - b = -1$ from $f(-1) = -8$.
- ✓ $a = 2$, $b = 3$.
- ✓ Quotient $x^2 + 3x + 6$.

c. ANSWER A · 4

i. The box is $16 - 2x$ long, $10 - 2x$ wide and x deep, so $x(16 - 2x)(10 - 2x) = 144$. Taking 2 out of each bracket: $4x(8 - x)(5 - x) = 144$, so $x(40 - 13x + x^2) = 36$, which is $x^3 - 13x^2 + 40x - 36 = 0$.

ii. $f(2) = 8 - 52 + 80 - 36 = 0$, so $x - 2$ is a factor. Synthetic division by 2 of 1, -13, 40, -36 gives 1, -11, 18, 0, and $x^2 - 11x + 18 = (x - 2)(x - 9)$. So $f(x) = (x - 2)^2(x - 9)$: $x = 2$ or $x = 9$.

A cut of 9 cm is impossible: two cuts of 9 cm are more than the card's 10 cm width. So the cut is 2 cm, and the box is 12 cm × 6 cm × 2 cm, which holds $12 \times 6 \times 2 = 144$ cm^3. ✓

MARKING SCHEME

- ✓ The equation formed from $x(16 - 2x)(10 - 2x) = 144$.
- ✓ $f(2) = 0$, so $x - 2$ is a factor.
- ✓ $f(x) = (x - 2)^2(x - 9)$, roots 2 and 9.
- ✓ 9 cm rejected (wider than half the card); the cut is 2 cm.

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6 Think

ANSWER HA · 2

No. If r and s are the rational roots, $(x - r)(x - s)$ divides the cubic and leaves a quotient of degree 1, $ax + b$, whose coefficients are rational (dividing numbers that are rational only ever gives rational numbers). Its root $-\frac{b}{a}$ is the third root, and it is rational. (A root such as $\sqrt{2}$ comes with its partner $-\sqrt{2}$.)

MARKING SCHEME

- ✓ Dividing by the two factors leaves a linear factor with rational coefficients.
- ✓ So the third root is rational: it can not be irrational.

7 Activity: Two ways to the same remainder

(not marked)

WHAT STUDENTS SHOULD FIND

a	$f(a)$	**Remainder**	**Factor?**
-2	0	0	yes
-1	8	8	no
0	6	6	no
1	0	0	yes
2	-4	-4	no
3	0	0	yes

The two columns always agree (the remainder theorem), and the three factors give $f(x) = (x + 2)(x - 1)(x - 3)$. At $a = 2$ the value is negative: between the roots 1 and 3 the curve dips below the X-axis.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	5	8	18	2	**33**

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